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**Formation of Structure  
in Dark Energy Cosmologies**

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ACADEMIC DISSERTATION

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# Abstract

Acceleration of the universe has been established but not explained. During the past few years precise cosmological experiments have confirmed the standard big bang scenario of a flat universe undergoing an inflationary expansion in its earliest stages, where the perturbations are generated that eventually form into galaxies and other structure in matter, most of which is non-baryonic dark matter. Curiously, the universe has presently entered into another period of acceleration. Such a result is inferred from observations of extragalactic supernovae and is independently supported by the cosmic microwave background radiation and large scale structure data. It seems there is a positive cosmological constant speeding up the universal expansion of space. Then the vacuum energy density the constant describes should be about a dozen times the present energy density in visible matter, but particle physics scales are enormously larger than that. This is the cosmological constant problem, perhaps the greatest mystery of contemporary cosmology.

In this thesis we will explore alternative agents of the acceleration. Generically, such are called dark energy. If some symmetry turns off vacuum energy, its value is not a problem but one needs some dark energy. Such could be a scalar field dynamically evolving in its potential, or some other exotic constituent exhibiting negative pressure. Another option is to assume that gravity at cosmological scales is not well described by general relativity. In a modified theory of gravity one might find the expansion rate increasing in a universe filled by just dark matter and baryons. Such possibilities are taken here under investigation. The main goal is to uncover observational consequences of different models of dark energy, the emphasis being on their implications for the formation of large-scale structure of the universe. Possible properties of dark energy are investigated using phenomenological parameterizations, but several specific models are also considered in detail. Difficulties in unifying dark matter and dark energy into a single concept are pointed out. Considerable attention is on modifications of gravity resulting in second order field equations. It is shown that in a general class of such models the viable ones represent effectively the cosmological constant, while from another class one might find interesting modifications of the standard cosmological scenario yet allowed by observations.

The thesis consists of seven research papers preceded by an introductory discussion.

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Tomi Koivisto  
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# List of publications

This thesis is based on the following papers.

**[1] The CMB spectrum in Cardassian models.**

With Hannu Kurki-Suonio and Finn Ravndal

Phys.Rev.D71:064027,2005 [astro-ph/0409163]

**[2] Growth of perturbations in dark matter coupled with quintessence.**

Phys.Rev.D72:043516,2005 [astro-ph/0504571]

**[2] Growth of perturbations in dark matter coupled with quintessence.**

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**[3] A note covariant conservation of energy momentum in modified gravities.**

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**[4] Cosmological perturbations in the Palatini formulation of modified gravity.**

With Hannu Kurki-Suonio

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**[5] Dark energy anisotropic stress and large scale structure formation.**

With David Mota

Phys.Rev.D73:083502,2006 [astro-ph/0512135]

**[6] The matter power spectrum in  $f(r)$  gravity.**

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**[7] Cosmology and Astrophysical Constraints of Gauss-Bonnet Dark Energy.**

With David Mota

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# Chapter 1

## Introduction

### 1.1 Dark energy: observations and theories

One of the most surprising findings of the end of the last century was that the universe seems to be accelerating. That was not because it wasn't realized that it was heading somewhere in the first place: the acceleration now refers to a speeding up of the universal expansion of the space. That expansion was already there in the equations Einstein in 1917 wrote to describe the universe as a whole. However, he added a constant, the so called  $\Lambda$ -term, to these equations in order to keep the universe static to comply with the prejudices of the time. The same year de Sitter was considering universes with only the cosmological constant but no matter at all. When the expansion of the universe was established in 1930 by Hubble's observations of the recession of distant galaxies, the  $\Lambda$  term was not needed anymore. Some decades later, cosmology was eventually becoming an exact science: observational data got more accurate, making it possible to test and falsify different theories about the universe in more detail. The main ideas that have survived until today are the Hot Big Bang and its extension at very early stages of the universe, the inflation. There still were some vague parts in the standard picture, like dark matter, something that was not seen but which was observed through its gravitational effects. However, about ten years ago an unexpected (to most, at least) change of the paradigm appeared compulsory, as the expansion of space was found to be accelerating. Einstein's  $\Lambda$ -term was vindicated - or de Sitters as well, since instead of the constant Einstein invoked to prevent matter from collapsing, the term required by acceleration will drive the universe into a state asymptotically devoid of any matter and thus identical to a space originally conceived by de Sitter. Already at the present, it seems, most of the energy density of the universe is due to the  $\Lambda$ -term, not matter.

To explain where this constant comes from is a great challenge to physics. The  $\Lambda$ -term corresponds to energy density residing in vacuum. It is perhaps counter-intuitive that such a term can be added consistently, since while the space expands, this vacuum energy density stays constant. The "weight of the vacuum" must then come with a pressure that is exactly equal to the energy density but with the opposite sign. In fact it is this negative pressure that exerts the effectively repulsive gravity that speeds up the expansion. In particle physics considerations the vacuums weight usually comes out nonzero. In principle the zero-point energy of any quantum field contributes to it. Though the sum of all the

contributions could be determined only from the theory of quantum gravity, one can estimate the energy density of the quantum vacuum,  $\rho_{VE}$ . The problem is that it comes out enormously larger than the cosmological observations would allow, as much as about 120 orders of magnitude larger than the  $\Lambda$ . Extreme fine-tuning would be needed for the contributions from different fields to cancel each other to this accuracy. The reason this expected vastness of  $\rho_{VE}$  became problematic with the observed acceleration is that earlier it was thought that some yet unknown symmetry principle might forbid vacuum energy altogether. It is considered far more aesthetic and reasonable to guess that  $\Lambda$  vanishes completely than to assume its value results from accidental cancellations with the accuracy of, say  $1/10^{120}$ . Therefore alternatives for the cosmological constant have been introduced. Some other energy component with large enough negative pressure might do as well, and then one would not have to resort to the cosmological constant. Generically, such an alternative is called dark energy. This could be some kind of exotic matter, manifestation of new theory of gravity or an effect of extra dimensions. What is common to all these alternatives is that some dynamics are associated to them, whereas the vacuum energy density does not evolve.

The question arises whether such dynamics are compatible with cosmological data. The acceleration was commonly established from the observations of distant supernovae[8, 9]. The luminosity versus redshift -relationship of these supernovae depends on the background expansion of the universe. Since the supernovae appeared to be dimmer than expected at given redshifts, implying they were farther away from us than in a universe filled with only matter, it was concluded that there is some energy component speeding up the expansion. If the energy component is not the  $\Lambda$ -term, the evolution of dark energy will result in variations in the expansion rate and thereby in the predicted luminosity-redshift -relationships of the supernovae. This is based on the fact that the supernovae of a specific type (classified as type Ia) seem to evolve nearly identically. For this reason these objects are suitable as so called standard candles. There are also other objects, such as certain types of galaxies, which can be useful as standard candles and hence as indicators of the evolution of the expansion rate at given redshifts. However, by considering the expansion history of the universe alone one cannot distinguish between different models of dark energy. In many, perhaps most of these models, a given evolution of the scale factor can be reproduced by tuning the parameters. As concrete examples, if dark energy is an effect of modified gravity, one may adjust the modification, or if it is due to potential energy of a scalar field, one might reshape the potential to generate exactly the same kind of expansion. Clearly, we need to take more physics into account.

The homogeneous expansion is of course an idealization; in reality the universe is filled with structures. It is widely believed that these originally formed from quantum fluctuations in the very early universe. After these quantum fluctuations somehow became classical during inflation, they have been growing for several billion years until the present. Initially small inhomogeneities have grown due to gravitational attraction. Matter collapses together and eventually forms smaller structures and then clusters of galaxies, empty voids appearing between<sup>1</sup>. In cosmology one is not interested in whereabouts of any particular overdense region, but in the overall structure, coarse-grained over different scales. As dark energy speeds up the expansion, matter cannot cluster as efficiently. Were

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<sup>1</sup>The acceleration could perhaps be related to forming of these structures, without introducing any dark energy. This possibility will briefly discussed in the section 3.2.4.

the dark energy just the cosmological constant, this would be the only effect. However, in general dark energy has also fluctuations. If there is any evolution in a cosmological fluid, it cannot be perfectly smooth. So it follows identically that a dynamical dark energy must have some non-trivial clustering properties. Inhomogeneities in dark energy then also interact gravitationally with matter, and thus they in principle have consequences to the distribution of matter. Since observations of this distribution at the present are consistent with the cosmological constant model, the predicted amount of clustering of dark energy should be small. This is usually the case for minimally coupled quintessence models, in which the dark energy resides in a very light scalar field. However, in models attempting to explain dark energy as a departure of Einstein's gravitational field equations, the departures can result in sometimes drastically unviable evolution of matter perturbations though the homogeneous universe would seem to expand in good agreement with observations.

The most abundant and accurate source of cosmological information comes from the cosmic microwave background (CMB) sky[10]. It consists of photons coming from the so called last scattering surface, which marks the era when the photons were released from their tight coupling to baryons and were let to travel through space. The difference in the temperature of these photons carry imprints of the primordial perturbations as they were set at the last scattering when the universe was about a thousand times smaller than today. At the same time, properties of the universe after last scattering can be deduced from the way the photons are affected during their travel to detectors. The angular fluctuations, which are of the order of  $10^{-5}$ , can be measured to remarkable accuracy, and the measurements agree with the predictions of usual cosmological models providing convincing support for inflation. Dark energy of course will modify the predicted CMB. An important effect comes already from the acceleration of average expansion: it pushes the last scattering surface further (since the thousandfold expansion has occurred slower than in a decelerating universe) and so there is a change in the overall geometrical properties. Also, the impact on matter perturbations as well as the possible dark energy fluctuations mentioned above can introduce variations in the redshifts of photons that travel through the gravitational wells due to overdensities. This is called the Integrated Sachs-Wolfe (ISW) effect. It is also interesting to correlate this ISW effect with the matter distribution, since the gravitational potentials depend on the evolution of matter perturbations. The nature of this dependence (given by the so called Poisson equation) depends on the theory of gravitation. For all these reasons it is useful and interesting to find out possible effects of dark energy on the CMB. One might in addition note that in the case that some amount of dark energy is present also at earlier times, there could be impact also on the primordial CMB spectrum forming before and at last scattering.

Having now reviewed the main motivations to introduce dark energy and the most promising possibilities to detect it, let us at this point make a small excursion into the question what this curious energy - if it is not just the vacuum constant - then might be in terms of more fundamental physics. Then it is useful to familiarize ourselves with the action principle. Due to the elegance of this principle, it is believed that any fundamental theory of physics should be specified by its action. The action principle, sometimes called the principle of least action or more appropriately, the principle of stationary action, asserts that the evolution of the physical fields involved is determined by the requirement that one number,  $S$ , sets to an extremal value. Technically, an action  $S$  is an integral of

some function of the fields involved, taken over the spacetime with the appropriate measure. Introducing small variations to the fields and requiring the resulting variation of the integral  $S$  to equal zero, a system of equations follows. These are the equations of motion that govern the evolution of the physical degrees of freedom such as the trajectories of the particles. In quantum mechanics extremization of the classical action does not determine the evolution of the system, which in principle depends on all imaginable trajectories. Then the action  $S$  is used to calculate the so called path integral giving the probability amplitudes of the various possible outcomes. Often the action integral  $S$  can be written in a compact form, from which one can readily read off many properties of the system like its possible symmetries.

The action of general relativity is an integral of the space-time curvature (the Ricci scalar), and of a function of the matter fields called the Lagrangian density for matter. When modelling dark energy, one introduces exotic matter fields and thus modifies this Lagrangian density. A different starting point is to consider only standard matter, but modify the gravitational part of the action. Then the predictions of general relativity, which have been highly successful to this date, are changed. However, there are motivations to proceed this way. Firstly, Einstein's theory of relativity has not been tested on cosmological scales, and so one might contemplate if the observed acceleration could be the first direct indication of our lack of understanding of gravity. The theory might indeed be modified in such a way that while the overall expansion of the universe might be altered, at smaller scales, like at our Solar system, the predictions of general relativity are retained to sufficient accuracy to comply with present experimental data. Secondly, general relativity is not held as a fundamental theory, since the gravitational field there has not been successfully quantized. As Einstein's equations relate this field to the matter, which fundamentally consists quantum fields, these equations have to be considered as some coarse-grained averages of an underlying quantum gravity. There are also unresolved issues concerning the nature of various singularities appearing in general relativity, such as black holes and the Big Bang singularity. However, in a conventional picture the corrections from quantum gravity are expected to be important only at very high energy scales (very small distances), whereas addressing the dilemma of dark energy would require deviations from Einstein's theory at low curvatures (distances of the order of present cosmological horizon).

String theory (or M-theory) has some prospects of possibly unifying all interactions and providing the relationships between quantum mechanics and general relativity[11]. In this framework the fundamental objects are not point-like particles but strings and the world has not four but ten (or why not eleven) dimensions when considered at tiny scales. These features could have also observable consequences at the low energy world we live in. There are indeed various dark energy models which make contact with string theory by beginning with a low energy string effective action (which might also be called a supergravity action). Though one has to bear in mind that such contact is tentative at the best, the speculative constructions based on string theory have provided various interesting and amusing possibilities to cosmology [12, 13, 14, 7, 15, 16]. Among the most prominent are brane cosmologies, where additional extra dimensions are not just compactified negligible but play a vital role[17]. Large compactified extra dimensions could be of the millimeter size and have gone unobserved; in an interesting case there in fact is an infinite, uncompactified fifth dimension as well [18]. Then matter (usually, at least) is considered to be confined to the four-dimensional brane (our universe) embedded in the higher dimensions,

but gravity is not. Thereby could perhaps some light be shed to the so-called hierarchy problem (that gravity is so weak compared to the other interactions) [19, 14]. The present acceleration might also be explained in such a context, since gravity could leak to the extra dimensions at large scales and thus be unable to decelerate the expansion of space [20].

Generically, the effective string action includes new fields and a series of corrections to the Einstein gravity. There appear scalar fields commonly referred to as moduli, which describe the compactified hidden dimensions. As they have to do with the internal geometry of extra dimensions, they do not usually directly couple to Einstein gravity. However, there is one exception, the scalar field that has been dubbed dilaton and that is associated with the overall size of the internal compactification manifold. The dilaton couples even to the four-dimensional space-time curvature. In addition to these scalar fields, string theory suggests contributions from higher-derivative curvature invariants than the space-time curvature. The leading order terms are quadratic, and in most versions of string theory they include the Gauss-Bonnet invariant. Interestingly, this invariant is the unique quadratic invariant that it is ghost-free (at least in de Sitter backgrounds) and that leads to second-order field equations. Actually, in four dimensions the Gauss-Bonnet invariant is a topological term and its contribution is classically trivial. However, this term could be coupled to the dilaton and then also affect the dynamics. It is often held that instead of the so called string frame the physical world is described by the so called Einstein frame, where the same action is rewritten with a metric where dilaton decouples from the scalar curvature. In the Einstein frame metric the Gauss-Bonnet term then acquires a dilatonic factor, and thus it might have cosmological effects[21, 7]. In the Einstein frame couplings with the matter Lagrangian and the scalar field can also be introduced, and such could be also useful in dark energy model-building[22, 23, 2].

Extensions of gravitational action are not unique to string theory, but come about for example in general Lovelock gravity[24] and in loop gravity[25]. The first is a generalization of the Gauss-Bonnet scheme, resorting only to curvature invariants with particularly favorable properties, and the latter is an approach towards unification of interactions with more general relativistic starting points. However, with the Gauss-Bonnet exception, higher order theories of gravity pose several theoretical and practical problems. Generally the field equations will involve derivatives of at least fourth order, which results in ghost and other kind of instabilities. Interestingly, in the so called Palatini formulation many of these problems are overcome. In the Palatini formulation the independent degrees of freedom in the gravitational action are reconsidered, and the resulting theory can then exhibit appealing features when applied to extensions of general relativity. This variational principle can be regarded as an alternative to the standard formulation (the so called metric one), and at least to our knowledge there is no physical principle stating which is the correct one. For the action of general relativity, both variational principles yield completely equivalent results. To summarize the possibilities in extending the gravity, one can allow scalar fields or additional curvature invariants in the action. Furthermore, one can couple the scalar with matter or the curvature, consider the action in the string or in the Einstein frame and apply the Palatini or the metric variation.

Fortunately, while there is no shortage of possibilities in explanations for the dark energy, one also has an increasing amount of data at hand to test these various models. Before letting the theory to run amok, one should find out the observational signatures of

different models and check whether they could be compatible with the present data. One of the best opportunities for this is the cosmic microwave background radiation (CMB) which has been measured with exquisite accuracy by the Wilkinson Microwave Anisotropy Probe (WMAP) satellite that published its long-awaited three-year results this year [26]. The microwave sky, will be measured to even greater precision by the Planck mission to be launched in couple of years from now. In addition, the galaxy distributions have been measured to high precision, most notably by the Sloan Digital Sky Survey (SDSS) [27] and the 2dF Galaxy Redshift Survey (2dFGRS) [28]. The observed matter distribution alone is sufficient information to exclude several ideas accounting for the present cosmic acceleration and for most of them, to tightly constrain their parameters. There are many other sources of cosmological information, but these, together with the luminosity-redshift relationships of the SNeIa, are the most precise ones at the present. By combining all the data, one could hope that it is eventually found out whether the dark energy is dynamical. If any evidence for any kind of evolution associated to dark energy is some day found to be compelling, after such a major discovery the hunt would begin for observational signatures peculiar to specific models among the plethora of possibilities, to nail down the basic properties of dark energy and establish its nature. If on the other hand dark energy is not observed to be dynamical, the problem is left, so we believe, to the particle physicist to explain the magnitude of the vacuum energy, and it would then remain to be seen for how long devising such dynamics to dark energy that would just escape detection remain popular entertainment in cosmology as the error bars about the cosmological constant keep shrinking.

## 1.2 Structure and contents of the thesis

The structure of thesis is the following.

- In section 2 we will review some basics of general relativity and its extensions.
- An introduction to cosmology is given in section 3, mainly in view of the contemporary dark energy problem.
- In the fourth section cosmology is discussed in a bit more detail: there linear perturbations are taken into account in order to make contact with CMB observations.
- In section V we attempt general descriptions of dark energy in terms of parameterizations.
- In section VI we consider some specific models of dark energy.
- Section VII contains a brief summary.

The two sections preceding the summary, which concentrate on the issue of how different alternatives for dark energy could be distinguished by their impact on linear perturbations, comprise the main content of the thesis. The price for generality of parameterizations is often a lack of predictiveness, while the applicability of model-specific calculations is usually limited. By combining these approaches we hopefully gain some insight about possible properties of what could be realistic dark energy.

This thesis is not ment to be a comprehensive review of different models of dark energy, but rather a discussion of some aspects (mainly related to perturbations) of accelerating cosmologies with references to a random selection of models (with the main emphasis on extensions of general relativity).

For a good and extensive survey of dark energy, see Ref.[29]; earlier reviews include Refs.[30, 31, 32, 33]; Ref.[34] is devoted to modified gravity in the metric formalism; Ref.[35] is a nice update on recent developments in the field and Refs.[33, 36] also introduce to the history of the cosmological constant problem.



## Chapter 2

# Gravity

Since the dynamics of the universe at large is governed by gravity, we begin with a review of the basics of general relativity. We will also discuss more speculative extensions of Einstein's gravity. Our aim is to describe the general structure of these theories.

### 2.1 General relativistic description of the universe

General relativity describes gravity as geometry of the spacetime. This geometry is determined by the matter content, while the movement of matter is in turn governed by the geometry. This interplay is encoded into the field equations, which read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M^2}T_{\mu\nu}. \quad (2.1)$$

Here  $M^{-2} = 8\pi G$  is the reduced Planck mass related to the Newton's constant  $G$ , which we will occasionally omit (working then in units  $M^2 = 1$ ). In the next subsections we will discuss more this fundamental relation, its derivation from an action principle and its generalizations. For now it suffices for us to recall that the Ricci tensor  $R_{\mu\nu}$ , as thus also its contraction, the curvature scalar  $R$ , is constructed from the metric  $g_{\mu\nu}$  in the following way. The Ricci scalar is the contraction  $R \equiv g^{\mu\nu}R_{\mu\nu}$  of the Ricci tensor

$$R_{\mu\nu} \equiv \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\alpha\lambda}^{\alpha}\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\mu\lambda}^{\alpha}\Gamma_{\alpha\nu}^{\lambda}, \quad (2.2)$$

where  $\Gamma$  is associated to the Levi-Civita connection of the metric (these  $\Gamma$ 's are then also called the Christoffel symbols),

$$\Gamma_{\beta\gamma}^{\alpha} \equiv \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}). \quad (2.3)$$

Thus the abovementioned objects named after Gregorio Ricci-Curbastro describe the geometry of the space in the combination defined as the Einstein tensor  $G_{\mu\nu}$ ,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \quad (2.4)$$

while the energy momentum tensor  $T_{\mu\nu}$  carries the information of the matter configuration. In four dimensions there are 16 field equations, but since in Einsteinian relativity

both the metric and  $T_{\mu\nu}$  are symmetric, the number of independent equations is reduced to ten. In general, this set of ten coupled nonlinear equations is practically impossible to solve. However, remarkable exceptions exist. These are spacetimes characterized by abundant symmetries. In such the number of independent degrees of freedom can be generously reduced, resulting in a simple enough system of equations of motion to be sometimes even analytically tractable. The first such case was discovered by Schwarzschild in 1915, and it describes a vacuum outside a spherically symmetric mass distribution.

The universe as a whole provides another example of a highly symmetric gravitational system. It's not static as the Schwarzschild case, but it's mass distribution couldn't be simpler. According to cosmological observations the Universe, when looked at large scales, is homogeneous and isotropic<sup>1</sup>. The most general metric for such spacetime is the Friedmann Robertson Walker (hereafter FRW) metric, for which the line element reads

$$ds^2 = a^2(\tau) \left\{ -d\tau^2 + \left[ \frac{dr^2}{1 - K r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \right\}. \quad (2.5)$$

Here  $a$  is the scale factor, which we normalize in such a way that today  $a_0$  equals unity. Here and hereafter we use subscript <sub>0</sub> to denote quantities evaluated at the present time. The conformal time  $\tau$  is related to the coordinate time  $t$  via  $d\tau = a(t)dt$ . The square brackets in Eq.(2.5) embrace the contribution to the line element from the spatial directions. When  $K > 0$  the spatial constant-time slices are positively curved, and when  $K < 0$ , these hypersurfaces are negatively curved. There is cosmological evidence that the curvature of the universe is negligibly small. Would there be a small but nonzero curvature, it would have to have grown detectable just recently, since the curvature evolves to dominate over the effect of matter to the expansion of the universe. A universe with  $K = 0$  is also what we except from the simplest models of inflation. Thereby we will mostly concentrate on flat models, considerably simplifying the analysis. FRW metric with  $K = 0$  is called flat, since then its space part has the Euclidean metric. In addition, the metric can then be written

$$ds^2 = a^2(\tau) [-d\tau^2 + \delta_{ij} dx^i dx^j], \quad (2.6)$$

using the Cartesian coordinates  $x_i$ . Thus the metric is also conformally flat, i.e. a Weyl transformation of the Minkowski metric  $diag(-1, 1, 1, 1)$ .

The isotropy of the Universe implies that it consists of matter which can be described as a perfect fluid. The energy momentum tensor of a perfect fluid is

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \quad (2.7)$$

and when evaluated in the comoving coordinates,

$$T_\nu^\mu = diag(-\rho, p, p, p), \quad (2.8)$$

where  $\rho$  is the energy density and  $p$  the pressure of the fluid. The homogeneity of the universe dictates that both  $\rho$  and  $p$  are functions of time only. The relation between them is called the equation of state,

$$w \equiv \frac{p}{\rho}, \quad (2.9)$$

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<sup>1</sup>If a spacetime is isotropic at every point (the cosmological principle), it is also homogeneous.

and the quantity  $w$  is called the equation of state parameter, which will denote as EoS in the following. We recall from thermodynamics that for relativistic matter  $w_r = 1/3$ , consistently with the field theoretical result that the energy momentum tensor for the Maxwell field is traceless. On the other hand, non-relativistic matter is approximately pressureless, and thus has zero EoS. For any matter at hand, the energy momentum is conserved. This means that

$$\nabla_\mu T^\mu_\nu = 0. \quad (2.10)$$

The  $\nu = 0$  component of this gives the continuity equation, from which we can deduce how the matter density evolves in an expanding universe,

$$\dot{\rho} + 3(1 + w)H\rho = 0. \quad (2.11)$$

An overdot denotes derivate with respect to (wrt) conformal time, and  $H$  is the conformal Hubble parameter,  $H \equiv \dot{a}/a$ . The solution is

$$\rho \propto a^{-3(1+w)}. \quad (2.12)$$

As expected, energy density in dust dilutes like one per the comoving volume as the universe expands. Energy density in radiation dilutes faster,  $\rho_r \propto a^{-4}$ , where the additional  $1/a$  can be explained by the loss of photon energy due to stretching of wavelength.

The relation of  $H$  to the energy content is given by the  $0 - 0$  component of the field Eq.(2.1)<sup>2</sup>, the Friedmann equation

$$H^2 + K = \frac{a^2}{3M^2}\rho. \quad (2.13)$$

The  $i - i$  component of the field equation does not yield additional information. With the knowledge of the properties of matter the universe consists of, i.e. given  $w$ , two unknowns remain,  $a$  and  $\rho$ , and these can be solved from the two Eqs.(2.11) and (2.13). This is true also the other way around. Given a measured expansion history, one can reconstruct the matter content.

## 2.2 Extensions of general relativity

The field equations of general relativity can be derived from the action principle. Hilbert discovered that a suitable form of the action is

$$S_H = \int d^4\sqrt{-g} \left[ \frac{1}{2}R + \mathcal{L}_m(g_{\mu\nu}, \Psi) \right], \quad (2.14)$$

where  $g$  is the determinant of the metric. The Lagrangian density  $\mathcal{L}_m$  depends on the metric and some matter fields  $\Psi$  and perhaps their first derivatives. Variation of the gravitational section with respect to the dynamical variable  $g^{\mu\nu}$  yields the Einstein tensor  $G_{\mu\nu}$ , defined

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<sup>2</sup>The tensor component  $G_{00}$  is calculated in the Appendix B. From there one also finds the connection coefficients, Eq.(B.1), which were used to derive the matter continuity equation. Similarly, we will hereafter use the results of Appendix B continuously (and mostly without explicit mention) in calculations involving metric variables.

as the LHS of the field equations (2.1), and one notes then that the energy momentum tensor on the RHS must be defined as the variation

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta(g^{\mu\nu})}. \quad (2.15)$$

We will stick to this definition throughout the thesis.

Although the action (2.14) is the simplest choice producing the observational successes of general relativity, no other a priori reason prevents from contemplating more general gravitational actions. A broad class of alternative gravity theories can be described in a unified way with the action

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2} f(R, \phi) + \mathcal{L}_\phi(g_{\mu\nu}, \phi, \partial\phi) + \mathcal{L}_m(g_{\mu\nu}, \Psi) \right], \quad (2.16)$$

including a general function  $f$  of the curvature scalar  $R$  and a scalar  $\phi$  with a Lagrangian density which is usually of the form

$$\mathcal{L}_\phi = -\frac{M^2}{2} \omega(\phi) (\partial\phi)^2 - V(\phi), \quad (2.17)$$

where we have denoted  $(\partial\phi)^2 \equiv (\nabla^\alpha \phi)(\nabla_\alpha \phi)$ . For a canonical scalar field the function  $\omega$  equals unity. The simplest examples of extended gravity is scalar-tensor theory, where  $f(R, \phi) = F(\phi)R$ . Such scalar tensor were originally introduced by Brans and Dicke to incorporate the Machs principle into general relativity. Before and after that different extensions of the Hilbert action have been considered. In all such cases, Einstein's gravity must be viewed as a limit of a hypothetical more general theory. In fact suggestions for such a theory can be found from fundamental physics. Quantization on curved spacetimes has been found long ago to require extension of the Einstein-Hilbert scheme by addition of higher-order curvature terms[37]. See the Introduction 1 for general discussion and Table 2.1 here for concrete examples of generalized gravity (similar tables can be found in Ref.[38]).

To analyze the structure of these theories, we will begin by showing that the generalized field equations still exhibit conservation of energy-momentum. Firstly, the equations of motion for the scalar field follow from setting the variation of the action with respect to scalar field to zero,

$$\frac{\partial \mathcal{L}_\phi}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}_\phi}{\partial (\nabla_\mu \phi)} = -\frac{1}{2} \frac{\partial f(R, \phi)}{\partial \phi}. \quad (2.18)$$

Using this result and differentiating then the energy-momentum tensor of the scalar field as defined in Eq.(2.15), it is straightforward to obtain

$$\nabla^\mu T_{\mu\nu}^{(\phi)} = -\frac{1}{2} \frac{\partial f(R, \phi)}{\partial \phi} \nabla_\nu \phi. \quad (2.19)$$

Varying the action Eq.(2.16) with respect to the metric we get the field equations that generalize now Eq.(2.1) to

$$F(R, \phi) R_{\mu\nu} - \frac{1}{2} f(R, \phi) g_{\mu\nu} = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) F(R, \phi) + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}, \quad (2.20)$$

Generalized grav.	$\frac{1}{2}f(R, \phi)$	$\mathcal{L}_\phi(\phi, \partial\phi)$	$p(R, \phi)$	$\varphi$	$\hat{V}(\varphi)$
Nonlinear [39]	$\frac{1}{2}f(R)$	0	$F(R)$	$\sqrt{\frac{3}{2}}\log(F)$	$\frac{RF-f}{2F^2}$
Quadratic [40]	$\frac{1}{2}(R + \alpha R^2)$	0	$1 + 2\alpha R$	$\sqrt{\frac{3}{2}}\log(F)$	$\frac{RF-f}{2F^2}$
CDTT [41]	$\frac{1}{2}(R - \mu^4/R)$	0	$1 + \mu^4/R^2$	$\sqrt{\frac{3}{2}}\log(F)$	$\frac{RF-f}{2F^2}$
Scalar-tensor [42]	$\frac{1}{2}F(\phi)R$	$L(\omega(\phi))$	$F(\phi)$	$\int \sqrt{\frac{\omega}{F} + \frac{3F'^2}{2F^2}} d\phi$	$\frac{V}{F^2}$
Brans-Dicke [43]	$\phi R$	$-\frac{\omega}{\phi}(\partial\phi)^2$	$\phi$	$\sqrt{\omega + \frac{3}{2}\log(\phi)}$	$\hat{V} = 0$
Dilatonic [44]	$\frac{1}{2}e^{-\phi}R$	$-\frac{1}{2}e^{-\phi}(\partial\phi)^2$	$e^{-\phi}$	$\sqrt{\frac{5}{2}}\phi$	$\hat{V} = 0$
NMC scalar [45]	$\frac{1}{2}(1 + \xi\phi^2)R$	$L(1)$	$(1 + \xi\phi^2)$	$\int \frac{\sqrt{1+\xi(6\xi-1)\phi^2}}{1-\xi\phi^2} d\phi$	$\frac{V}{1-\xi\phi^2}$
Conformal [46]	$\frac{1}{2}(1 + \frac{1}{6}\phi^2)R$	$L(1)$	$(1 + \frac{1}{6}\phi^2)$	$\sqrt{6}\tanh^{-1}\frac{\phi}{\sqrt{6}}$	$\frac{V}{1-\frac{1}{6}\phi^2}$
Induced [47]	$\frac{1}{2}\epsilon\phi^2R$	$L(1)$	$\epsilon\phi^2$	$\sqrt{6 + \frac{1}{\epsilon}\log\phi}$	$\frac{V}{\epsilon\phi^2}$
GR + $\phi$ [Tab. 3.1]	$\frac{1}{2}R$	$L(1)$	1	$\phi$	$V$

Table 2.1: Some interesting actions that generalize the Einstein gravity. There  $L(\omega) = -\frac{1}{2}\omega(\phi)(\partial\phi)^2 - V(\phi)$ .

where  $F(R, \phi) \equiv \partial f(R, \phi)/\partial R$ . Note that since the Ricci scalar involves second derivatives of the metric, fourth order derivatives appear in the first term in the RHS of Eq.(2.20). Thus these non-linear gravity theories (non-linear refers to dependence of  $f$  on  $R$ ) are examples of fourth-order gravity theories. We saw that the scalar field energy momentum is not always conserved. Are the generalized field equations Eq.(2.20) nonetheless consistent with the simple equality Eq.(2.10)? The answer turns out to be positive, which can be traced to the fact that our matter fields  $\Psi$  that enter into the Lagrangian  $\mathcal{L}_m$  are minimally coupled to gravity, i.e. decouple from the function  $f$ . From now on we lighten the notation by keeping the dependence of  $f$  and  $F$  on  $R$  and  $\phi$  implicit. Taking the covariant divergence on both sides of Eq.(2.20) yields  $n$  equations

$$\begin{aligned}
(\nabla^\mu F)R_{\mu\nu} + F\nabla^\mu R_{\mu\nu} - \frac{1}{2} \left[ F\nabla^\mu R + \frac{\partial f}{\partial \phi} \nabla^\mu \phi \right] g_{\mu\nu} \\
= (\square \nabla_\nu - \nabla_\nu \square)F + \nabla^\mu T_{\mu\nu}^{(\phi)} + \nabla^\mu T_{\mu\nu}^{(m)}. \quad (2.21)
\end{aligned}$$

These simplify by using Eq.(2.19) and the definition of  $G_{\mu\nu}$ :

$$(\nabla^\mu F)R_{\mu\nu} + F\nabla^\mu G_{\mu\nu} = (\square \nabla_\nu - \nabla_\nu \square)F + \nabla^\mu T_{\mu\nu}^{(m)}. \quad (2.22)$$

On purely geometrical grounds,  $\nabla^\mu G_{\mu\nu} = 0$  and  $(\square \nabla_\nu - \nabla_\nu \square)F = R_{\mu\nu} \nabla^\mu F$ . These identities follow from the definitions of the tensors  $G_{\mu\nu}$  and  $R_{\mu\nu}$  [48]. Therefore  $\nabla^\mu T_{\mu\nu}^{(m)} = 0$ , and the conservation energy-momentum in fourth order  $f(R, \phi)$ -gravities is confirmed.

Recently there has been interest in models where a function of the curvature scalar enters into the action to multiply a matter Lagrangian. These have been introduced in view of possible mechanism for dynamical relaxation of the vacuum energy [49, 50]. Such terms were not included in our action (2.16), but we consider them briefly here as an

example of cases where the covariant energy-momentum conservation might be violated. Since for that purpose the form of those functions do not matter, we set  $f = R$  and  $\omega(\phi) = V(\phi) = 0$  for simplicity. Thus we write the action as

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2} R + k(R) \mathcal{L}_m(g_{\mu\nu}, \Psi, \dots) \right]. \quad (2.23)$$

The field equations are then

$$G_{\mu\nu} = -2K \mathcal{L}_m R_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) 2\mathcal{L}_m K + k T_{\mu\nu}^{(m)}, \quad (2.24)$$

where  $K \equiv dk/dR$  and  $R$ -dependence is again kept implicit. Remembering the definition in (2.15) and proceeding as previously, one now finds that

$$k \nabla^\mu T_{\mu\nu}^{(m)} = (\nabla^\mu R) \left[ g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}^{(m)} \right] K. \quad (2.25)$$

If  $k$  is a constant or if  $\delta \mathcal{L}_m / \delta g_{\mu\nu} = 0$ , the covariant divergence of the energy-momentum tensor vanishes. Otherwise the matter fields must satisfy equations of motion which are equivalent to (2.25).

### 2.2.1 Conformal frames

It is possible to recast nonlinear gravity in a form of Einstein gravity and an interacting scalar field. This is done by acting on the metric by a suitable conformal transformation; the interested are referred to Appendix A for technical details and to Ref.[51] for a review of the use of these transformations in the literature. Such transformation is a local change of scale, and in practice it is performed by multiplying the metric with a coordinate dependent function. Written in terms of a conformally rescaled metric, the Jordan frame action (2.16) (defined by the minimally coupled matter part) is transformed into an action formally representing a different theory of generalized gravity. The rescaled metric includes dependence on a scalar degree of freedom, and introduces violation of the conservation laws similar to Eq.(2.25).

Change of frame can be viewed as a reparameterization. Therefore actions which are conformal transformations of each other, indeed can present the same physical theory, though then the different sets of variables (i.e. frames) are operationally different (i.e. they are related to observable quantities in different ways). For a concrete example, consider the metric (2.5). There the scale factor  $a$  can be measured via the redshift  $z$ , since  $z = 1/a - 1$ . This is what we will mean when referring to "the physical metric" in the following. Now, by performing a conformal transformation, one can write a line-element equivalent to Eq.(2.5), but featuring a rescaled scale factor (and correspondingly rescaled coordinates). This other scale factor then does not have the same conventional relation to the observed redshift that we quoted above. It is this what we have in mind when we state that conformal frames "are not physically equivalent". As mentioned, they can be used to describe the same physics given a correct interpretation for the transformed variables. In our example, we can obtain the measured redshift from the transformed scale factor by retransforming back into the original frame. We see that this is simple, but also that it is worthwhile to explicate what is precisely meant by the physical inequivalence of the conformal frames. Then a question seems arise that given a theory, which among the conformally related

(and theoretically consistent) frames is the physical one. However, simply enough again, without the physical frame singled out, a theory is incomplete. In such a case the most suitable frame should be decided by observations.

To see this, we will elaborate a bit the interpretation of the transformation. It is useful to write the conformal factor in terms of a scalar field. By choosing this factor suitably, the gravitational part of the action can be recast into exactly the Hilbert form. This is called the Einstein frame. In this way it is possible to consider higher orders in the Jordan frame modified field equations to be represented by the scalar field in the Einstein frame<sup>3</sup>. Now the scalar field enters into the matter action, which thus becomes non-minimally coupled. Only in the special case of so called conformal matter (which has traceless energy momentum tensor) the coupling remains minimal, since for such matter the action is proportional to the zeroth power of the metric and so the scalar field dependence is cancelled away from the matter action. Now consider a measurement of the gravitational mass of an object by comparing to the weight of another object. In the Einstein frame, the gravitational mass depends on the scalar field, but since the dependence is the same for all objects (which consist of ordinary matter), it cancels out and the measured ratio should be the same everywhere. However, if one determines the inertial mass of a particle by for example measuring the force it exerts on a string, a different result can in principle be found at different regions of spacetime, because the scalar field (and thus the conformal factor) is space and time dependent. So if one sets the Einstein frame units in one point in such a way that the inertial and the gravitational masses are equal, they are so for all particles, but in principle only in that particular point. Elsewhere they are not. We deduce that in the Einstein frame the equivalence principle is violated. One can, however, perform calculations in the Einstein frame but still consider the Jordan frame as physical. Then the results should be retransformed back into the Jordan frame in order to compare with observations. For example, the conformal factor involving the scalar field then disappears from the expressions for particle masses and the equivalence principle is retained in the Jordan frame. It is sometimes considered that matter is minimally coupled to gravity in the physical frame[39], and that one can and should take advantage of this fact when determining which of the conformally equivalent metrics is the physical one. While this is not an unreasonable assumption, it is neither a compelling argument. Violations of the equivalence principle can be theoretically consistent and compatible with observations.

To recapitulate, in an extended theory of gravity of the form (2.16), one generically finds a dynamical effective gravitational constant. By transforming to the Einstein conformal frame, a constant force of gravity is recovered, but the masses of particles are found to evolve in time. The frames are mathematically equivalent, since the equations and their solutions in the two frames are the same, simply written in terms of different variables. It is also clear that the frames are physically inequivalent, since they describe very different phenomena. Both of these cases are mathematically self-consistent, and thus there is no a priori reason to exclude either. One could say that the two frames represent different theories, if a theory is understood as a specification of the physical variables and an action written in terms of them. In different a terminology a theory just equals an action, but

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<sup>3</sup>Fourth order gravity introduces two extra derivatives in the Jordan frame, and these are represented by the scalar and its derivatives in the Einstein frame. A higher order gravity results in additional scalar fields; for example, a sixth order gravity originating from terms like  $\square f(R)$ , is turned into a double-scalar theory in the Einstein frame.

then (most of) the physics is left unpredicted by a theory.

## 2.3 The Palatini variation

Once the gravitational action is nonlinear in  $R$ , the question which variational principle to apply becomes relevant. The Palatini variation<sup>4</sup> of a nonlinear gravity action leads to a different theory than the standard (metric) variation we applied in the previous subsection. As was seen, the metric variation of extended gravity theories result in fourth order differential equations which are difficult to analyze in practice. The Palatini formulation, in which the connection is treated as an independent variable is more tractable than the metric one, and it can also in general exhibit better stability properties, since it yields the modified field equations as a second-order differential system. Mathematical convenience does not of course prove that the Palatini variation would be the fundamentally correct procedure. However, this possibility might be interesting also according to some theoretical prejudices. The second-order nature of the Palatini formulation is conceptually more reconcilable with better-known physics than the metric alternative, where the action in the beginning contains second derivatives of the metric, and in the end one has to specify initial values up to third derivatives to predict the evolution of the system (except in general relativity, of course). The doubling of the variational degrees of freedom in the Palatini formulation has an analogy with the Hamiltonian mechanics, where the coordinates and momenta of particles are treated as independent variables[48]. On the more speculative side, it is also interesting that the Palatini scheme of gravity can be recovered in unification of general relativity with topological quantum field theory[52].

When the connection is promoted to an independent variable (we will call it  $\hat{\Gamma}_{\beta\gamma}^\alpha$ ), The Ricci tensor can be defined without referring to the metric at all,

$$\hat{R}_{\mu\nu} = \hat{\Gamma}_{\mu\nu,\alpha}^\alpha - \hat{\Gamma}_{\mu\alpha,\nu}^\alpha + \hat{\Gamma}_{\alpha\lambda}^\alpha \hat{\Gamma}_{\mu\nu}^\lambda - \hat{\Gamma}_{\mu\lambda}^\alpha \hat{\Gamma}_{\alpha\nu}^\lambda, \quad (2.26)$$

We define then the Ricci tensor as  $R \equiv g^{\mu\nu} \hat{R}_{\mu\nu}$ . Then the function  $f$  in the action is regarded as a function the metric, the connection, and the scalar field,

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2} f(R(g_{\mu\nu}, \hat{\Gamma}_{\beta\gamma}^\alpha), \phi) + \mathcal{L}_\phi(g_{\mu\nu}, \phi, \partial\phi) + k(R(g_{\mu\nu}, \hat{\Gamma}_{\beta\gamma}^\alpha)) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right] \quad (2.27)$$

Note that we consider here a slightly more general action than in Eq.(2.16), since there is a possible non-minimal coupling also in the matter sector. The field equations got by setting variation with respect to the metric to zero seem simple,

$$F \hat{R}_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + 2K \mathcal{L}_m \hat{R}_{\mu\nu} = T_{\mu\nu}^{(\phi)} + k T_{\mu\nu}^{(m)}. \quad (2.28)$$

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<sup>4</sup>This is also called first order formalism, since there only first derivatives of the dynamical variables appear in the Hilbert action. This name would perhaps also be more proper, since Attilio Palatini was not first to apply it to a gravitational action, but Einstein who published his results concerning this variational principle in 1925. Nevertheless, Palatini had wrote similar equations in the context of electrodynamics. -This principle could also be called the metric-affine principle. However, the resulting theory is not necessarily a metric-affine theory of gravity, as will be clarified below.



However, now  $R$  and  $\hat{R}_{\mu\nu}$  are not the ones constructed from the metric. By varying the action (2.27) with respect to  $\hat{\Gamma}_{\beta\gamma}^\alpha$ , one gets the condition

$$\hat{\nabla}_\alpha \left[ \sqrt{-g} g^{\beta\gamma} (F + 2K\mathcal{L}_m) \right] = 0, \quad (2.29)$$

where  $\hat{\nabla}$  is the covariant derivative with respect to  $\hat{\Gamma}$ , implying that the connection is compatible with the conformal metric

$$h_{\mu\nu} \equiv (F + 2K\mathcal{L}_m)^{2/(n-2)} g_{\mu\nu} \equiv \omega^{2/(n-2)} g_{\mu\nu}. \quad (2.30)$$

However, the connection  $\hat{\Gamma}_{\beta\gamma}^\alpha$  is not the physically interesting connection on the manifold, just as the metric  $h_{\mu\nu}$  does not have any direct physical content. It just governs how the tensor we call  $\hat{R}_{\mu\nu}$  appearing in the action settles itself in order to minimize the action. One could also consider the case that the metric  $h_{\mu\nu}$  is the measurable, but that would lead to freely falling particles following geodesics that are not those corresponding to the metric  $g_{\mu\nu}$ . This would lead to a different theory, which could also be considered, but will not concern us for now<sup>5</sup>. We will return to these discussions in sections 2.3.2 and 6.3.

As the Ricci tensor is constructed from the metric  $h_{\mu\nu}$ , the easiest way to find it in terms of  $g_{\mu\nu}$  is to use a conformal transformation. We get

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \frac{(n-1)}{(n-2)} \frac{1}{\omega^2} (\nabla_\mu \omega)(\nabla_\nu \omega) - \frac{1}{\omega} (\nabla_\mu \nabla_\nu \omega) - \frac{1}{(n-2)} \frac{1}{\omega} g_{\mu\nu} \square \omega. \quad (2.31)$$

Note that the covariant derivatives above are with respect to  $g_{\mu\nu}$ . The curvature scalar and Einstein tensor follow straightforwardly,

$$R = R(g) - \frac{2(n-1)}{(n-2)} \frac{1}{\omega} \square \omega + \frac{(n-1)}{(n-2)} \frac{1}{\omega^2} (\partial \omega)^2, \quad (2.32)$$

where  $R(g)$  is the corresponding scalar constructed from the metric  $g_{\mu\nu}$ ,

$$\hat{G}_{\mu\nu} = G_{\mu\nu} + \frac{(n-1)}{(n-2)} \frac{1}{\omega^2} (\nabla_\mu \omega)(\nabla_\nu \omega) - \frac{1}{\omega} (\nabla_\mu \nabla_\nu \omega - g_{\mu\nu} \square \omega) - \frac{(n-1)}{2(n-2)} \frac{1}{\omega^2} g_{\mu\nu} (\partial \omega)^2 \quad (2.33)$$

and a somewhat more tedious calculation<sup>6</sup> then gives an identity that will soon come in handy to us,

$$\nabla^\mu \hat{G}_{\mu\nu} = -\frac{(\nabla^\mu \omega)}{\omega} \hat{R}_{\mu\nu}. \quad (2.34)$$

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<sup>5</sup>A somewhat related issue is that if the matter Lagrangian depends on a connection, it must be specified whether this connection is the Levi-Civita one or the one which the Palatini variation yields. The latter choice would lead to a complicated theory of the Dirac field, but according to Ref. [53] this would be the natural choice. However, no apparent reason was given why it would be more unnatural to couple matter fields to the Levi-Civita connection of  $g_{\mu\nu}$  as usually. This assumption to which we restrict ourselves in this section is perfectly consistent and simplifies the structure of the theory a lot. Nevertheless, we mention that to allow the independent connection to enter the matter Lagrangian would open up, in addition to some difficulties perhaps (for example there will be some arbitrariness in making the matter sector "projectively invariant"), interesting possibilities for theories[54, 55].

<sup>6</sup>One can arrive at this result by relating the  $g$ -divergence of  $\hat{G}_{\mu\nu}$  to its vanishing  $h$ -divergence via the difference of the corresponding connection coefficients, or alternatively by taking directly the  $g$ -divergence of Eq.(2.33).

Having now the field equations in their complete form, we see that they again involve second derivatives of  $R$ , coming about from Eqs.(2.31) and (2.32), and this  $R$  is supposed to be of second order. However, we can make an interesting observation at this point. From the trace of the field equation (2.28),

$$(F + 2K\mathcal{L}_m)R - \frac{n}{2}f = T^{(\phi)} + kT^{(m)}, \quad (2.35)$$

we get an algebraic relation between  $R$  and the matter variables. We call this central relation the structural equation. For the moment, say we have no scalar field, the coupling  $k$  with dust ( $T^{(m)} = -\rho_m$ ) is minimal,  $k = 1$ , and the function  $f$  is a power law  $f = c_0 R^\alpha$ . It follows that  $R/c_0 = (\rho/(n/2 - 1))^{1/\alpha}$ . Such a solution we can then insert into the field equations (2.28): therefore they are now of the second, not of fourth order in derivatives.

Let us then finally check the conservation law. Taking now the divergence of the field equations (2.28) similarly as in the previous case, we get

$$\begin{aligned} (\nabla^\mu \omega) \hat{R}_{\mu\nu} + \omega \nabla^\mu \hat{R}_{\mu\nu} - \frac{1}{2} \left[ (\omega - 2K\mathcal{L}_m) \nabla_\nu R + \frac{\partial f}{\partial \phi} \nabla_\nu \phi \right] \\ = \nabla^\mu T_{\mu\nu}^{(\phi)} + k \nabla^\mu T_{\mu\nu}^{(m)} + (\nabla^\mu k) T_{\mu\nu}^{(m)}. \end{aligned} \quad (2.36)$$

This simplifies, by using Eqs.(2.19) and (2.34), to

$$k \nabla^\mu T_{\mu\nu}^{(m)} = (\nabla^\mu R) \left[ g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}^{(m)} \right] K. \quad (2.37)$$

These conditions are formally the same as the ones found in the metric formulation, Eq.(2.25), but here  $R$  is given by Eq.(2.32). Thus the divergence of the matter energy-momentum tensor again vanishes identically when  $k$  is a constant. If matter is nonminimally coupled to curvature (i.e.  $K \neq 0$ ), we have  $n$  constraints which the matter fields must satisfy. These are satisfied identically in the special case that  $\partial \mathcal{L}_m / \partial g^{\mu\nu} = 0$ . Otherwise the non-minimal curvature coupling influences the matter continuity non-trivially.

### 2.3.1 Noether variation of the action

These results may be understood as the generalized Bianchi identity as given by Magnano and Sokolowski[39]. It is straightforward to generalize their derivation of this identity to  $n$  dimensions, include scalar field couplings in the gravitational action and apply the Palatini variational principle, but we outline the procedure here for completeness. Especially the incorporation of the independent connection  $\hat{\Gamma}$  in this derivation might not be immediately clear[56, 57].

Consider an infinitesimal point transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon \xi^\mu, \quad (2.38)$$

where  $\xi^\mu$  is a vector field vanishing on the boundary  $\partial\Omega$  of a region  $\Omega$ . The fields entering into the gravitational action are shifted such that  $f(x) \rightarrow f(x')$ . Since the gravitational action is extremized in the classical solution, we demand that the action (2.27) is invariant

to first order in the infinitesimal parameter  $\epsilon$  under the transformation (2.38),  $\delta S = 0 + \mathcal{O}(\epsilon^2)$ , where:

$$\delta S = \int_{\Omega} d^n x \left( \frac{\delta[\sqrt{-g}(\frac{1}{2}f + \mathcal{L}_{\phi})]}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\delta[\sqrt{-g}(\frac{1}{2}f + \mathcal{L}_{\phi})]}{\delta \hat{\Gamma}_{\beta\gamma}^{\alpha}} \delta \hat{\Gamma}_{\beta\gamma}^{\alpha} + \frac{\delta[\sqrt{-g}(\frac{1}{2}f + \mathcal{L}_{\phi})]}{\delta \phi} \delta \phi \right). \quad (2.39)$$

So we let here  $f$  depend on the metric and its derivatives up to any order  $m$ . Note that for the metric formalism action (2.16)  $m = 2$ , whereas for the Palatini formalism action (2.27)  $m = 1$ . Then one applies the Gauss theorem  $m$  times and drops the boundary terms to arrive at

$$\begin{aligned} 2\sqrt{-g}Q_{\mu\nu} &\equiv \frac{\delta(\sqrt{-g}f)}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \\ &\equiv \frac{\partial(\sqrt{-g}f)}{\partial g^{\mu\nu}} - \partial_{\alpha} \frac{\partial(\sqrt{-g}f)}{\partial g_{,\alpha}^{\mu\nu}} + \dots + (-1)^m \partial_{\alpha_1} \dots \partial_{\alpha_m} \frac{\partial(\sqrt{-g}f)}{\partial g_{,\alpha_1 \dots \alpha_m}^{\mu\nu}}. \end{aligned} \quad (2.40)$$

Here the important caveat is that we assume all the derivative terms up to  $m$ 'th order to vanish in the boundary  $\partial M$ . There might be some subtleties involved with this condition when  $m > 1$  (see Ref.[58] a recent discussion and references therein), but now we just take for granted that it holds. In fact this was done already in the previous subsection when we wrote the resulting order field equations in the metric formalism. This possible problem does not appear in the Palatini formalism discussed in this subsection, and in general for an action involving just the variables  $g_{\mu\nu}$ ,  $\hat{\Gamma}$ , and possibly their first derivatives. In the metric formalism the second term in Eq.(2.39) is identically zero since there is no independent connection variable, and in the Palatini formulation the extremization of the action with respect to the variations in the connection  $\hat{\Gamma}$  guarantees the vanishing of the second term. Similarly, the equation of motion for the scalar field states nothing but that the third term disappears. Since under the transformation (2.38) the metric transforms as

$$g^{\mu\nu} \rightarrow g^{\mu\nu} + \epsilon \nabla^{(\mu} \xi^{\nu)}, \quad (2.41)$$

we then get, using the symmetry  $\mu \leftrightarrow \nu$  and the definitions of Eqs.(2.15), (2.40) that

$$\begin{aligned} \delta S &= 2\epsilon \int_{\Omega} d^n x \sqrt{-g} (Q_{\mu\nu} - T_{\mu\nu}^{(\phi)}) \nabla^{\mu} \xi^{\nu} \\ &= 2\epsilon \int_{\partial\Omega} \sqrt{-g} (Q_{\mu\nu} - T_{\mu\nu}^{(\phi)}) \xi^{\nu} dS^{\mu} - 2\epsilon \int_{\Omega} d^n x \sqrt{-g} \nabla^{\mu} (Q_{\mu\nu} - T_{\mu\nu}^{(\phi)}) \xi^{\nu} = 0 \end{aligned} \quad (2.42)$$

The surface term is zero since  $\xi^{\nu}$  vanishes at the boundary. Since it is otherwise arbitrary,  $\nabla^{\mu} Q_{\mu\nu} = \nabla^{\mu} T_{\mu\nu}^{(\phi)}$ . The energy conservation follows from the field equations since they read  $Q_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)}$ . Thus the matter action can be considered to be separately invariant under the point transformations.

### 2.3.2 Conformal and geodesic structure

In the Palatini formulation the conformal metric (2.30) inevitably appears also when considering an extended gravity in the Jordan frame. According to the least action principle,

the gravitational section in the action (2.27) settles in such a way that a function of the contraction of the Ricci tensor associated with the conformal metric  $h_{\mu\nu}$  is extremized. This Ricci tensor has the usual relation to parallel transport<sup>7</sup> in the manifold, but now to the parallel transport according to the connection of the Einstein conformal metric. Therefore the resulting field equations are different from both standard general relativity and its extensions when considered in the metric formalism. In the Palatini formulation the response of gravity to matter features, effectively, additional sources in the matter sector. Still, the equations of motion for matter take the same form in all of these cases. The energy-momentum conservation laws are the same. The conformal metric  $h_{\mu\nu}$  plays an interesting mathematical role in the Palatini formulation of the action principle, suggesting a possible derivation from and an interpretation within a more fundamental framework of quantum gravity. However, from the viewpoint of the resulting classical theory of gravitation we are now considering, the metric  $h_{\mu\nu}$  and the connection  $\hat{\Gamma}$  associated with it can be regarded as just auxiliary fields which were used in the formulation of the action principle to derive the field equations.

The caveats here are 1) the assumption that the independent connection does not enter into the matter action and 2) that one might also in the Palatini formalism switch to the Einstein frame where the metric  $h_{\mu\nu}$  is physical. We discuss the latter briefly. The two frames are not symmetric, which is manifest in the fact that the metric  $g_{\mu\nu}$  continues to have some relevance in the Einstein frame: although the metric we measure there is  $h_{\mu\nu}$ , also there the continuity and the geodesics of matter are given by  $g_{\mu\nu}$ . In this sense we might say that Einstein frame theory furnishes a bi-metric structure, since there the measured metric is  $h_{\mu\nu}$ , but some aspects of the motion of matter are associated with the other metric  $g_{\mu\nu}$ . As an example, consider geodesic motion of particles. A geodesic is a curve which parallel transports its own tangent vector. If particle with velocity  $u_\mu$  moves along its geodesic, then  $u_\nu \nabla^\nu u_\mu = 0$ . These same curves also extremize the proper time of the particles along their path, and the so called geodesic hypothesis states that free particles indeed move along such curves. Now the conservation of matter energy-momentum tensor and also the geodesic hypothesis, is violated in the Einstein frame. Note that this happens in the metric formulation as well, and that such bi-metricity arises in any other conformally equivalent frame except the Jordan one. Thus the Palatini variational principle leads to nothing new when geodesics are concerned.<sup>8</sup>

The metric and geodesic structures cannot be arbitrarily decoupled, since the free fall of particles is uniquely determined by the equations of motion for matter. These in turn, as shown in detail above, follow from the field equations which can be written solely in terms of the metric. Thus the laws governing the motion of particles are inscribed to the field equations, although they are occasionally introduced as a seemingly independent postulate. As a concrete example, consider dust. Then we can write  $T_{\mu\nu} = \rho u_\mu u_\nu$ , where  $\rho$  is the energy density and  $u_\mu$  the  $n$ -velocity of the fluid. The covariant conservation then

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<sup>7</sup>A vector field  $V$  on a smooth curve  $\gamma(t)$  is parallel if  $\nabla_{\gamma(t)} V = 0$  with any  $t$ . Parallel transport of a vector  $V$  around a parallelogram spanned by coordinates  $\mu$  and  $\nu$  is (in a torsionless spacetime) proportional to contraction of Riemann tensor and  $V$ . We mean that  $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma$ .

<sup>8</sup>This has not been very clear in the literature. It has been stated that the Palatini formulation of extended gravity theories are bi-metric in the sense that their metric structure is determined by  $g_{\mu\nu}$ , while their geodesic structure would be given by the conformally equivalent metric  $h_{\mu\nu}$  [59, 60, 61, 62]. To highlight the difference of this notion to ours: 1) In the Einstein frame, we find exactly the opposite roles for the two metrics, 2) In the Jordan frame, the same metric  $g_{\mu\nu}$  assumes both of the roles.

gives

$$\nabla^\mu T_{\mu\nu} = u_\nu \nabla^\mu (\rho u_\mu) + \rho u_\mu \nabla^\mu u_\nu = 0. \quad (2.43)$$

Because  $u_\mu u^\mu = -1$ , we find by multiplying this equation with  $u_\nu$  that  $\nabla^\mu (\rho u_\mu) = 0$ . Therefore Eq.(2.43) reduces to  $u_\mu \nabla^\mu u_\nu = 0$ , which is nothing but the statement that the dust particles follow the geodesics of the metric  $g_{\mu\nu}$ . This already shows that geodesics in general cannot be determined by the  $h_{\mu\nu}$ -compatible connection  $\hat{\Gamma}$  that appears in the action 2.27. The generalization of the above derivation of the geodesic equation from the covariant conservation of matter energy-momentum to the case of an arbitrary body of sufficiently small size and mass turns out to be much less trivial, and for this more general case we refer the reader to [63].

# Chapter 3

## Cosmology

We will review the nowadays standard cosmology, the so called  $\Lambda$ CDM model. It has been said that this otherwise fine model has two major problems, the first being the  $\Lambda$  and the second the CDM. We outline these problems and various approaches that have been applied to attack them.

### 3.1 The contents of the universe

Given an energy component  $i$ , it is conventional to define its density relative to the critical density  $\rho_{crit} \equiv 3M^2H^2/a^2$  as

$$\Omega_i = \rho_i / \rho_{crit}, \quad (3.1)$$

Taking into account all possible contributors to the total energy budget, one can write the Friedmann Eq.(2.13) as

$$1 - \frac{K}{H^2} = \sum_i \Omega_i = \Omega_{total}. \quad (3.2)$$

This is why the density  $\rho_{crit}$  is called the critical. When it equals the total density, the universe is flat. In open universes, the total density is smaller, in closed universes larger than the critical. We observe a cosmological background of black body radiation at a temperature of about 2.7 K, and see galaxies of luminous matter about us. So we have at least photons and so called baryonic matter contributing to the energy density. There are also theoretical reasons to believe that a neutrino background exists, although they cannot be directly observed. Neutrinos may have tiny masses[64, 65], but in this thesis we will consider them as relativistic particles. Therefore we can, in many occasions, lump them together with photons and write for example,  $\Omega_r \equiv \Omega_\gamma + \Omega_\nu$ . From the scaling of radiation energy density one quickly infers that at an early stage of the universe the relativistic matter must have been a more significant contributor, i.e.  $\Omega_\gamma \gg \Omega_b$ , although now the radiation has cooled down and  $\Omega_b \gg \Omega_r$ .

However, a considerable amount of cosmological and astrophysical evidence supports a picture in which the aforementioned constituents account only for less than 1/20 of the total energy density in the present universe. The rest resides in something not described by the standard model of particle physics.

That there is more matter than can be seen was first inferred from observations of dispersion relation of the Coma cluster. Existence of dark matter halos about galaxies is clearly indicated by the observed rotation velocities of galaxies as a function of the distance from the center of the galaxy. The rotation velocity stays constant far beyond the optical radius of a galaxy. It is possible to estimate the amount of baryonic matter in the universe from astrophysical measurements, and on the other hand to calculate the predicted primordial abundance of light elements produced at the big bang nucleosynthesis (BBN) taking place during the first minutes of the universe. There is some discrepancy between the two values thus obtained, but both imply  $\Omega_b \leq 0.05$ . This is also consistent with the observations of CMB anisotropies[66].

### 3.1.1 Dark matter

From the cosmological viewpoint, there are two main categories of nonbaryonic dark matter. Light particles, with masses around  $m \leq 30$  eV would be relativistic at decoupling, and could constitute hot dark matter (HDM). Since dark matter should be weakly interacting, neutrinos with small mass would be ideal examples of HDM. However, the pressure in such matter would suppress the formation of cosmological large scale structure. It seems that a pure HDM scenario is not cosmologically viable. In a cold dark matter (CDM) scenario, the dark matter particle is more massive. Such matter can be approximately pressureless, producing a good agreement with the measurements of large scale structure. One possibility would seem to be a considerable amount of massive and nonluminous objects of baryonic matter residing in galactic halos in form of e.g. brown dwarfs and planets, but this guess is in disagreement, in addition to the BBN prediction of  $\Omega_b$ , with microlensing observations. Therefore one must look for candidates of dark matter particles from extensions of the standard model of particle physics. These particles would have to be sufficiently weakly interacting to have escaped detection in modern particle detectors. For example, axion which arises in the solution of the CP problem, has been considered as a well motivated candidate for the CDM particle. Axions would be very light particles, but generally one looks for a WIMP (weakly interacting massive particle). Supersymmetric extensions of the standard model provide the nowadays most prominent CDM candidates. The particle CDM consists of could be the lightest supersymmetric partner, if it is stable (which would be the case when R-parity is conserved) or at least has a lifetime of cosmological scale.

Since dark matter is observed only through its gravitational effects, it can be contemplated whether one could do without it by modifying suitably the laws of gravity[67]. In the so called MOND (modified Newtonian dynamics) scheme, a new scale is introduced to the Newton gravity:

$$\mu(|\mathbf{a}|/a_0)\mathbf{a} = -\nabla\Psi_N, \quad a_0 \approx 10^{-10}ms^{-2}. \quad (3.3)$$

The function  $\mu(x)$  is equal to one for large accelerations,  $x \gg 1$ , so as to comply with Solar system and laboratory experiments, but  $\mu(x) = x$  when  $x \ll 1$ . Due to the latter property, such an ansatz can successfully explain the rotation curves of galaxies, but since it is basically non-relativistic, for example gravitational lensing is difficult to address. The MOND law (3.3) violates energy conservation, but a generalized law has been found using the Lagrangian formalism which overcomes this problem. It was only recently that this

formalism was developed both relativistic and covariant[68]. This TeVeS (Tensor-Vector-Scalar) theory is set up by demanding that matter is living in a different metric than the  $g_{\mu\nu}$  we construct our Hilbert action from. The physical metric  $\bar{g}_{\mu\nu}$  is related to the Einstein metric by a generalized conformal transformation, a "disformal transformation"

$$\bar{g}_{\mu\nu} = e^{-2\phi}(g_{\mu\nu} + \mathcal{U}_\mu \mathcal{U}_\nu) - e^{2\phi} \mathcal{U}_\mu \mathcal{U}_\nu, \quad (3.4)$$

where  $\phi$  is a scalar field and  $\mathcal{U}_\nu$  a vector field. Actions have been prescribed to these fields which govern their dynamics. The TeVeS theory can deal with gravitational lensing situation unlike the non-relativistic MOND, but dynamics of clusters might not be explained in TeVeS either without appeal to dark matter. It seems that to reconcile all observations indicating existence of dark matter with modified gravitational dynamics necessitates intricate, perhaps epicyclic, devices.

### 3.1.2 The cosmological constant

In 1997 it was found that the expansion of the universe is accelerating. Specifically, this surprising result was inferred from the observed luminosity-redshift relations of type Ia supernovae (SNIa) [8, 9]. The SNIa supernovae are believed to have nearly identical absolute magnitudes. If the luminosity of a bright object at redshift  $z$  is  $L$ , the flux we observe from that object is

$$F = \frac{L}{4\pi S^2(\tau(z))(1+z)^2} \equiv \frac{L}{4\pi d_L^2}, \quad (3.5)$$

where  $d_L$  is called the luminosity distance, and  $S(x) = x$  in a flat universe,  $S(x) = \sinh(\sqrt{-\Omega_K}x)/\sqrt{-\Omega_K}$  in a negatively curved and  $S(x) = \sin(\sqrt{\Omega_K}x)/\sqrt{\Omega_K}$  in positively curved. Here  $\tau(z)$  is the conformal time that has elapsed between redshift  $z$  and the present. This arises because the total luminosity through a spherical shell in a comoving coordinates is constant. The area of the shell is  $4\pi S^2(\tau)$ . The luminosity scales as energy of the emitted photons multiplied by the number of photons passing through the shell in a unit time, giving  $\sim a \cdot a = a^2 = 1/(1+z)^2$ . Now magnitudes are related to fluxes via  $m = -(5/2) \log(F) + \text{constant}$ . Therefore measuring the apparent magnitude of an object with known absolute magnitude at a given redshift  $z$  gives the conformal distance to  $z$ . Actually the absolute brightness of SNIa is not well established, but since it is supposed to be the same for all individual supernovae, measuring their apparent luminosities at different redshifts gives information of the expansion history. Since the distant supernovae appear to be fainter than they would in a matter dominated universe, it is concluded that the universe is dominated by a component which drives the acceleration and thus must have a large negative pressure. From recent SNIa data[69] one can infer that the transition from deceleration to acceleration occurred at about<sup>1</sup>  $z \approx 0.5$ . If one assumes the cosmological constant  $\Lambda$  to be the agent of acceleration and  $K = 0$ , the data indicate that the contribution of vacuum energy to the critical density is about  $\Omega_\Lambda \approx 0.7$  today. This is also consistent with the matter power spectrum of large scale structure as inferred from

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<sup>1</sup>Though one should be aware that such results depends strongly on underlying assumptions. With rather minimal assumptions of only homogeneity and isotropy, the evidence from supernova data does not compel to accept but a constant acceleration[70].



galaxy redshift surveys like the Sloan Digital Sky Survey (SDSS)[71] and the 2dF Galaxy Redshift Survey (2dFGRS)[72], and the anisotropies in the CMB [66].

However, we lack a theoretical explanation for the observed value of the cosmological constant. Generically, one could expect the existence of universal vacuum energy. Given a generic field, there is no principle stating that its zero-point energy should vanish. In richer theories there may be such principles, such as supersymmetry or conformal invariance. The observed world, however, shows no sign of such symmetries, so they must be severely broken if they exist at all. It is not unreasonable to assume that an unknown symmetry cancels vacuum energy away altogether. Otherwise we can only roughly estimate that in a quantum theory of gravity the vacuum energy density would be given by

$$\rho_\Lambda \propto \int_0^\infty \sqrt{k^2 + m^2} k^2 dk, \quad (3.6)$$

if one would account for the contribution of all the scales. Since this integral is ultraviolet divergent, one might assume that the fundamental theory is such that very high energy scales do not contribute, and introduce a cut-off to the integral. A natural scale would be  $\sim M_P$ . Then we expect  $\rho_\Lambda \sim M_P^4$ . But this is enormously large compared to the value  $\Omega_\Lambda \rho_{crit}$ , the discrepancy being about 120 orders of magnitude. Having a lower cut-off for  $k$ , motivated by for example a guess for the supersymmetry breaking scale, would give a smaller value for the theoretical vacuum energy, but still (at least) tens of orders of magnitude away from the observationally allowed values. Extreme fine-tuning would be required to get the observed cosmological constant from field theory.

Another, related fine-tuning issue is the so called cosmic coincidence problem. It is simply that the  $\Omega_\Lambda \sim \Omega_m$  just at the present. This is a disturbing fact since before now, the scale factor has grown (roughly) by a factor  $10^{10}$  since say the BBN era with  $\Lambda$  being completely negligible, but after the scale factor has grown by say a factor of ten from now, everything but the  $\Lambda$  will be negligible. Looked at this way, the transition era seems almost instantaneous, and we would like to understand why it takes place right now. Actually, how long the acceleration has been ongoing is not so relevant aspect of the coincidence. When plotted with the cosmic time as the  $x$ -axis, the picture is not evocative of a coincidence at all, since the time from  $z \sim 1$  to the present, the era of dark energy dominance this far, spans about the half of the age of the universe, several billion years. The rub is there that, since the energy density in matter decreases during the cosmic evolution, it has necessarily been initially hugely larger in the early universe than the constant vacuum energy.

### 3.2 Alternative explanations

With no satisfactory derivation of the cosmological constant, numerous alternative cosmologies have been proposed where  $\Lambda = 0$ . According to the assumptions underlying these alternatives, we categorize these scenarios into three classes. In general relativity, an accelerated isotropic and homogeneous spacetime must be filled by matter with dominantly negative pressure. However, it is not necessarily a constant. Indeed, the most popular replacement of  $\Lambda$  is a dynamical component with negative pressure today. To do without negative pressures at all, one might relax the assumption that the universe is well described by the homogeneous FRW equations. Yet one possibility is to accept that the universe is

dominated by homogeneously distributed dust-like matter and the coarse-graining applies as conventionally. Then gravity at large scales cannot be described by general relativity.

### 3.2.1 Quintessence

Generically, the effects of dynamical dark energy to the background expansion are fully described by its EoS (as a function of the redshift). Any observational evidence for  $w_{de} \neq -1$  or  $\dot{w}_{de} \neq 0$  would prove that the dark energy is something else than vacuum energy. Observational limits[73] on the EoS (if it assumed a constant) are  $-1.34 < w_{de} < -0.79$  and on its variation (assuming a simple parameterization)  $dw/dz = 1.0^{+1.0}_{-0.8}$ . A more recent analysis using various distance indicators gives tighter limits[74],  $w = 0.96 \pm 0.08$  or  $w = 0.87 \pm 0.1$  (depending on the set of supernova observations) for dark energy with constant EoS. The cosmological constant provides a good fit, but the data allows more general models. It also seems to marginally favour models of so called phantom dark energy (at least does not exclude them), which exhibit supernegative EoS,  $w_{de} < -1$ . In the following, we will mention a couple of most prominent dark energy fluid candidates.

The most popular candidate for a dynamical dark energy is a light scalar field [75, 76] generically called quintessence. Recall that the cosmological energy density and pressure of a canonic scalar field  $\phi$  are given by

$$\rho_\phi = \frac{1}{2a^2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2a^2}\dot{\phi}^2 - V(\phi). \quad (3.7)$$

If the field is slowly rolling, i.e. its potential energy dominates,  $V(\phi) \gg a^{-2}\dot{\phi}^2/2$ , the EoS turns out negative. One can look for effective scalar fields from for example supersymmetric field theories and from string/M-theory.

From the cosmological point of view, quintessence models can be classified according to the shape of their potentials. Nowadays the field must be situated in a sufficiently flat region of its potential for the slow roll condition to apply. Further restrictions to the shape of the potential can be imposed if one requires a so called tracker property. This implies that there exists an attractor solution to which the field is drawn to from a wide range of differing initial conditions. This is of course preferable in order to avoid fine tuning of the initial conditions. Scaling of the quintessence energy density in the tracking phase is determined by the background expansion of the universe. Therefore  $w_\phi = w_\phi(w_B)$ , where  $w_B$  is the background EoS, and for the field to "track" the background fluid, one requires that  $w_\phi$  is also close to  $w_B$ . For an exponential potential,  $V(\phi) \sim e^{\lambda\phi}$ , the tracking is "exact", meaning that  $w_\phi = w_B$ . Thus these potentials are not very good quintessence candidates, since the field either is not in the tracking phase (fine-tuning is necessary), or its effect on the expansion just merges into the background (then at present  $w_\phi = 0$ ). This is unfortunate since exponential potentials frequently appear in high energy physics considerations.

This has motivated various modifications to the simplest exponential scenario. An example is the Planck scale quintessence, where a polynomial prefactor is introduced to change the slope of the potential (see table 3.1). One advantage of this form is that now the parameters  $A$ ,  $B$ ,  $\phi$  and  $\lambda$  can all be (roughly) of the order of the Planck scale. This is not possible for the simplest quintessence fields. An inverse power-law potential has also been considered as a quintessence candidate. However, it has been found that the form

Quintessence model	Coupling	Kinetic	Potential
Exponential [76]	1	$X$	$V_0 \exp(-\lambda\phi)$
Inverse power [76]	1	$X$	$V_0/\phi^\alpha$
Supergravity [83]	1	$X$	$V_0 \exp(\lambda\phi^2)/\phi^\alpha$
Planck scale [84]	1	$X$	$V_0[A + B(\phi - \phi_0)^2]e^{-\lambda\phi}$
Cosmon [75]	$\exp[\beta(\phi - \phi_0)]$	$X$	$V_0 \exp(-\lambda\phi)$
Dilaton-derived [23]	$1 + \lambda(\phi - \phi_0)^b$	$X$	$V_0 \exp(-\lambda\phi)$
K-essence [85]	1	$K(\phi)p(X)$	0
Phantom [86]	1	$-X$	$V(\phi)$
Tachyon [87]	1	$V(\phi)\sqrt{1-X}$	0
Ghost condensate [88]	1	$-X + V_0 e^{\lambda\phi} X^2$	0

Table 3.1: A selection of quintessence models. If the field is non-minimally coupled to the curvature, as in extended quintessence [79], the model falls into scalar-tensor categories of Table 2.1.

inspired by supergravity corrections is both observationally (it results in more negative  $w_\phi$ , in better accordance with the SNIa data) and theoretically (one expects corrections in line with this from supergravity since  $\phi = \mathcal{O}(M_P)$ ) preferred to the simple power-law form of the potential.

In order to onset the acceleration after the scaling era, one might also couple an exponential quintessence to matter[2] or to curvature[7]. In a scalar-tensor theories of gravity the latter possibility is realized [77, 78, 14]. In extended quintessence models [79] one can alleviate the fine-tuning because of the effect of "R-boost" on the tracking behaviour [79, 80], and find other interesting modifications to standard quintessence phenomenology, for instance in the clustering properties[81] or implications for weak lensing observations [82].

The quintessence scenario has also been extended to cases of noncanonical scalar fields. Specifically, using a nonstandard kinetic term, it is possible to consider called phantom dark energy or general k-essence models. For the canonical quintessence,  $-1 < w_\phi < 1$ , but when the sign of the kinetic term is flipped, it is seen from Eq.(3.7) that also  $w_\phi < -1$  is possible. Generically, models in which the kinetic term of the scalar field is a more general function of the field derivatives, are called k-essence.

### 3.2.2 Quartessence

An idea that has gained attention is to lump all unknown physics into the dark sector by assuming a single fluid accounts for both dark matter and dark energy. Thus this fluid must resemble cold dark matter in the earlier universe, whereas it should exhibit large negative pressure nowadays. Since this reduces the number of cosmic energy components to four, these models could be called quartessence. In table 3.2 we list some proposals for the unified energy density of the dark sector. The first eight parameterizations are just specific cases of the Modified polytropic Cardassian (MPC) model, and the following four are its modifications. The last one corresponds to the leaking gravity model of Dvali, Gabadadze and Porrati (DGP), in which the modified Friedmann equation can be derived

Modified polytropic Cardassian	$[Aa^{3q(\nu-1)} + Ba^{-3q}]^{\frac{1}{q}}$	[89]
New Generalized Chaplygin gas	the same	[90]
The $\Lambda$ CDM limit	$q = 1$ and $\nu = 1$	[1]
Cardassian expansion	$q = 1$	[91]
Polytropic Cardassian	$\nu = 1$	[89]
Generalized Chaplygin gas	$\nu = 2$	[92]
Variable Chaplygin gas	$q = 2$	[93]
Chaplygin gas	$\nu = 2, q = 1$	[94]
Modified Chaplygin gas	$(A + Ba^{-3})^q$	[95]
Exponential Cardassian	$(Aa^{-3} + B) \exp[(\frac{qB}{Aa^{-3}+B})^\nu]$	[96]
Extra dimension inspired	$Aa^{-3}[1 + \exp(-Ba^{-3})]^q$	[97]
Phenomenological approach	$A(1 + Ba^{-1})^{q-\nu}[1 + Ca^{-\nu}]$	[98]
Leaking gravity (DGP)	$Aa^{-3} + B - \sqrt{B^2 + ABa^{-3}}$	[20]

Table 3.2: Energy densities in funny models of unified dark matter and dark energy.

from a set-up where our universe is embedded in an infinite fifth dimension. The rest are perhaps more phenomenological parameterizations.

The Cardassian model has been loosely motivated by guesses for an effective energy density in the observable universe in the presence of bulk matter, though attempts to construct a suitable higher-dimensional energy momentum tensor have not succeeded. For both MPC and DGP the effective energy density is then described by a modified Friedmann equations and reduces to the formulas in Table 3.2 only when the universe is dominated by dust. In principle there would be modifications also in the earlier eras, and at the present the contribution from radiation would be effectively different from the standard  $\rho_r \sim a^{-4}$ . However, these effects are very small and thus formulas in the table are in any case good approximations. In fact a fluid interpretation has been proposed for the MPC expansion[99], which then features exactly the EoS implied by the first entry in the table.

The prototype for unified models is the Chaplygin gas, with the simple equation of state  $p = -A/\rho$ . As an example of a brane construction where the Chaplygin gas equation of state can appear consider the following setting [16]. Denoting the bulk metric  $g_{MN}$  and the the coordinates  $X^M$ , the induced metric on the brane in general can be written [100]

$$\bar{g}_{\mu\nu} = g_{MN}(X(x))\partial_\mu X^M \partial_\nu X^N, \quad (3.8)$$

with  $\mu$  running over the brane indices. In the case of one compactified extra dimension of size  $\ell_5$ , and the scalar field  $0 \leq \theta(x^\mu) \leq \ell_5$  describing the embedding, the coordinates then being  $(x^\mu, \theta(x))$ , it is easy to see that the induced metric could take the form

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \theta_{,\mu}\theta_{,\nu}. \quad (3.9)$$

The action for the brane taking into account only the tension  $t$  is thus<sup>2</sup>

$$S_{brane} = \int -d^4x \sqrt{-\bar{g}} t = - \int d^4x \sqrt{-g} \sqrt{1 - g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}} t. \quad (3.10)$$

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<sup>2</sup>The second equality follows from the identity  $\det(a_{ij} - b_i b_j) = \det(a_{ij}) b^m (1 - a^{-1})_{mn} b^n$ .

Then varying with respect to  $g_{\mu\nu}$  the definition (2.15) yields

$$T_{\mu\nu} = t \left( \frac{\theta_{,\mu}\theta_{,\nu}}{\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}} - g_{\mu\nu}\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}} \right). \quad (3.11)$$

Now, with the identification  $u_\mu = \theta_{,\mu}/\sqrt{g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$ , we see that Eq.(3.11) is an energy momentum tensor of a perfect fluid obeying the Chaplygin gas equation of state when  $t^2 = A$ .

Alternatively, the unified fluid can be taken as an energy component without referring to extra dimensions. As an example, we construct a scalar field that behaves like a MPC fluid. The same procedure could be applied to find a scalar field that generates any arbitrary background expansion. By the conservation law (2.10) we see that the pressure in a MPC fluid is

$$p = \frac{1}{q} \left[ (1 - \nu - q)(A + Ba^{3(\nu-1)})^{\frac{1}{q}} + (\nu - 1)A(A + Ba^{3(\nu-1)})^{\frac{1}{q}-1} \right]. \quad (3.12)$$

From Eq.(3.7) we find that

$$\dot{\phi}^2 = a^2(\rho + p) = \frac{(1 - \nu)Ba^{3(\nu-1)}}{q(A + Ba^{3(\nu-1)})^{1-\frac{1}{q}}}. \quad (3.13)$$

By integrating the square root of this equation we get the solution for the scalar field as a function of  $a$ :

$$\phi(a) = \pm \lambda \tanh^{-1} \sqrt{\frac{A}{B}a^{3q} + 1}, \quad (3.14)$$

where we have defined the constant  $\lambda = \frac{2}{\sqrt{3q(1-\nu)}}$ . Since Eq.(3.7) tells that  $V(\phi) = (\rho - p)/2$ , we find straightforwardly the potential our scalar field lives in:

$$V(\phi) = \frac{A^{\frac{1}{q}}}{2q} (\cosh^2(\lambda\phi))^{\frac{1}{q}-1} [2q \cosh^2(\lambda\phi) + (\nu - 1) \sinh^2(\lambda\phi)]. \quad (3.15)$$

Furthermore, it is easy check that for a tachyon field (see table 3.1) in a constant potential  $V_0$ , the pressure and energy density are connected by the simple Chaplygin gas equation of state with the identification  $A = V_0^2$ .

### 3.2.3 Gravitational dark energy

As opposed to energetics of unknown fluids, the cosmic speed-up could stem from modifications to general relativity. It has been suggested that the gravitational action may be an effective higher dimensional action on our brane in such a way that gravity is weaker at cosmological scales. Another approach is to begin with a four-dimensional extended theory of gravity as discussed in the previous section. In either way, it is possible that the cosmic expansion is not given by the standard Friedmann equation, but that a dust-dominated universe exhibits an accelerating expansion. Since the gravitational field equations have not been experimentally tested at so vast scales, they might indeed deviate from the Einstein theory at cosmological distances.

Here as a specific class of gravitational alternative for dark energy we will consider non-linear gravity in the first-order formalism. Consider late cosmology where  $\rho = \rho_m = \rho_c + \rho_b$  and thus  $p = 0$ . The modified Friedmann equation then follows from the field equation (2.28) with Eqs.(2.31) and (2.32),

$$\left(H + \frac{1}{2} \frac{\dot{F}}{F}\right)^2 = \frac{1}{6F} (\kappa\rho + f). \quad (3.16)$$

By using the conservation of matter (2.10) and proceeding as in [101], we find that the Hubble parameter may be expressed solely in terms of  $R$ ,

$$H^2 = \frac{1}{6F} \frac{3f - RF}{\left[1 - \frac{3F'}{2F} \frac{RF - 2f}{RF' - F}\right]^2}. \quad (3.17)$$

For a known function  $f(R)$ , one can solve  $R$  at a given  $a$  from the structural relation (2.35) and thus get the expansion rate Eq.(3.17) by just algebraic means<sup>3</sup>.

In order to consider these models quantitatively, we must specify the form the gravitational Lagrangian. Let us use the two parameterizations

$$A: \quad f(R) = R - \alpha R^\beta, \quad B: \quad f(R) = R - d e^{-bR} \quad (3.18)$$

where  $\alpha$  is positive and has dimensions of  $H^{2-2\beta}$ , while  $d$  is also positive with dimensions of  $H^2$ . The (dimensionless) exponent  $\beta$  is less than unity and the parameter  $b$  is positive, lest the correction to the Einstein-Hilbert action would interfere with the early cosmology. Given  $\beta$  (or  $b$ ) and the amount of matter in the present universe,  $\Omega_m$ , determine the scale  $\alpha$  (or  $d$ ). Not all combinations of  $\alpha$  and  $\beta$  (or  $d$  and  $b$ ) are consistent with a flat matter dominated universe today, and therefore we rather use  $\Omega_m$  and  $\beta$  ( $\Omega_m$  and  $b$ ) as our pair of parameters. To demonstrate that models defined by the Lagrangian (3.18) can produce a plausible expansion history, one can compare the predicted evolution of the Hubble parameter, Eq.(3.17) to cosmological data. As an example we compute the CMB shift parameter, which quantifies the conformal distance to the last scattering surface and thus tells about the apparent position of the peaks in the CMB spectrum. It is given by

$$\mathcal{R}_{shift} = \sqrt{\Omega_m} H_0 \int_0^{z_{dec}} \frac{dz'}{H(z')}, \quad (3.19)$$

where  $z_{dec}$  is the redshift at decoupling. For these parameters we use best-fit values found from the CMB analysis of the WMAP team [66],  $\mathcal{R}_{shift} = 1.716 \pm 0.062$  and  $z_{dec} = 1088_{-2}^{+1}$ . We plot the constraints resulting from fitting the CMB shift parameter in Fig. 3.1. Projecting the left panel to the  $(\alpha, \beta)$  -plane would reproduce Fig. 1 of [101], where are presented also further constraints from background expansion derived from multiple data sets.

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<sup>3</sup>When considering perturbations, we will also need the derivatives  $\dot{H}$  and  $\ddot{H}$ . Expressions for these can be derived similarly. However, it is unnecessary to report the resulting (rather lengthy) formulae here, since when resorting to numerical analysis one may equally well evaluate numerically the derivatives of Eq.(3.17), the algebraic formulae providing just a means of a consistency check.

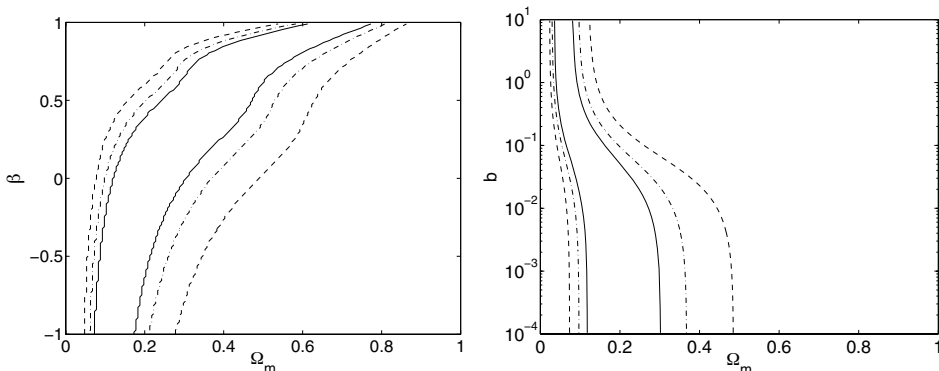


Figure 3.1: The 68, 90 and 95.4 % confidence contours arising from fitting the CMB shift parameter. The left panel corresponds to the parameterization A and the right panel to the parameterization B.

### 3.2.4 Dark energy from backreaction

Some claims have been made that superhorizon perturbations, generated by a non-standard phase in the inflation, could produce the observed acceleration[102]. Many problems have been pointed out in any such mechanism[103], the general opinion being that such is not possible.

Subhorizon perturbations seem more promising. In the standard CDM cosmology, perturbations at small scales have grown non-linear during the evolution of galaxies and other structure in the universe. To associate the onset of acceleration with the formation of sizable inhomogeneities would be a nice way to address the coincidence problem [104, 105]. However, there are yet conceptual and practical obstacles in determining whether the small-scale perturbations could affect the overall expansion of the background universe. Yet there is no compelling reason to expect that perturbed FRW would not be an accurate description of the universe at whole, but the issue is interesting and certainly worthwhile to study as it tries out a basic assumption underlying the common approach to cosmology.

When one adopts the FRW description of the world, it is assumed that the volume-averaged energy momentum tensor and the metric of the universe is adequate for its description on cosmological scales. To prove that this is legitimate, one would have to first find the metric that is solution in the inhomogeneous universe, and then show that its volume average is indeed the FRW metric. However, in practice this is overwhelmingly difficult, and thus there is room for speculation about the possibility that the inhomogeneities of the universe cause significant deviations from the predictions based on the simple FRW description. Such an impact of small scales to the overall expansion is called backreaction. So the proposition is based on the claim that since Einstein equations are nonlinear, the Friedmann equation, involving quantities that have been smoothed prior to inclusion, might not look the same as the equation obtained by averaging the full equation including all the inhomogeneous quantities<sup>4</sup>, the difference being an effective dark energy

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<sup>4</sup>A bit more accurate is to say that the time evolution and averaging do not commute.

density[106]. In appreciating the plausibility of this backreaction, it is essential to realize that its significance in general is determined by the fraction of space occupied by non-linear regions rather than the size of the individual regions. Thus one might retain the postulate that an overall scale factor well describes the universe, but discard the conventional Friedmann equation as the law of evolution of this scale factor. More quantitatively, if we are smoothing over a domain  $D$ , the Friedmann equation is [107]

$$H_D^2 = \frac{1}{3M^2} \langle \rho \rangle_D - \frac{1}{6} (\langle R \rangle_D + Q_D), \quad (3.20)$$

where in addition to the averaged energy density,  $\langle \rho \rangle_D$ , there is the backreaction variable  $Q_D$  and the averaged spatial curvature  $\langle R \rangle_D$  contributing to the expansion inside the domain,  $H_D$ . The effective pressure is then proportional to  $-Q_D$ , and thus, if  $Q_D$  would come with the right magnitude and the positive sign, the backreaction could give rise to an acceleration. To derive Eq.(3.20), one has to assume that the fluid is irrotational (there is no vorticity), otherwise considerable complications arise. The backreaction variable is given by the shear  $\sigma^2$  and the variance of the expansion rate  $\theta$ ,

$$Q_D \equiv \frac{2}{3} (\langle \theta^2 \rangle_D - \langle \theta \rangle_D^2) - 2 \langle \sigma^2 \rangle_D. \quad (3.21)$$

The shear appears already in the exact inhomogeneous field equations, but the variance arises in the averaging procedure. It is this variance that might accelerate the average expansion rate (shear tends to decelerate it), although everywhere the local expansion rate would be decelerating. The impact of collapsing dust to the expansion rate through the increasing variance is physically interpreted to be due to the increase of the relative volume of the regions of space which are expanding faster. However, it is perhaps not yet entirely clear whether it then is the average expansion rate<sup>5</sup> one probes with, say, the supernova observations [108]. In any case, a more quantitative study is needed to uncover if this suggestion is compatible with cosmological data, see Refs.[109, 110] for reviews.

Another approach is to assume from the beginning that the universe is better described by an inhomogeneous metric and then to compute whether ostensible acceleration occurs if the measurements are interpreted in terms of the homogeneous standard model [111]. This approach has been used in the context of the Lemaitre-Tolman-Bondi model, where radial dependence is allowed in an isotropic universe [112], so the average density can vary with its distance to us. The Lemaitre-Tolman-Bondi metric can be written as

$$ds^2 = -dt^2 + \left( \frac{\partial R(r, t)}{\partial r} \right)^2 \frac{dr^2}{1 - k(r)} + R^2(r, t) d\Sigma_{(2)}^2, \quad (3.22)$$

where  $d\Sigma_{(2)}^2$  is the metric of a two-dimensional sphere. A desirable feature of the metric (3.22) is that for it one can easily calculate the background expansions and luminosity distances, given an ansatz for the (spherically symmetric) distribution of matter. With this additional freedom (as compared to the FRW case) due to arbitrary  $r$ -dependences<sup>6</sup>, it is

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<sup>5</sup>Implicitly defined in Eq.(3.20) as  $a(t) \equiv \left( \frac{\int d^3x \sqrt{g^{(3)}(t, \mathbf{x})}}{\int d^3x \sqrt{g^{(3)}(t_0, \mathbf{x})}} \right)^{\frac{1}{3}}$ . The  $g^{(3)}$  here is the determinant of the metric of spatial hypersurfaces.

<sup>6</sup>If  $R(r, t)$  is separable, the FRW case is recovered.



not surprising that with a suitable density profile for the matter distribution one can indeed find a good fit with the supernovae data without including dark energy [108]. However, then one should abandon the Copernican principle and to comply with the observed isotropy of the CMB, place Earth to the central region covering about one millionth of the volume of the perfectly spherical bubble [113]. It is not clear whether other observations, like those of LSS (the homogeneous distribution of luminous matter, the baryon oscillation scale) could be reproduced in this model. In addition, the question how such a bubble appears has not been addressed, and so the link to the era of structure formation present in the backreaction context mentioned above seems a little bit lost in this approach at the present.

# Chapter 4

## Perturbations

### 4.1 Gauges

In the homogeneous and isotropic world of Friedmann, Robertson and Walker the line-element was given by

$$ds^2 = a^2(\tau) \left[ -d\tau^2 + g_{ij}^{(3)} x^i x^j \right], \quad (4.1)$$

where the three-metric  $g_{ij}^{(3)}$  determines the geometry of the spatial hypersurfaces. In a flat universe it is just  $g_{ij}^{(3)} = \delta_{ij}$ , and from Eq.(2.5) one can read off a possible form of  $g_{ij}^{(3)}$  for a non-flat case. When considering inhomogeneities, one clearly cannot use the simple metric Eq.(4.1). However, since the universe seems to be roughly homogeneous, one can still benefit from the simplicity of the metric Eq.(4.1) by using it as a background, and describing the inhomogeneities as small perturbations about this background. Ten independent components must be introduced for the most general inhomogeneous metric (since the symmetries discussed in section 2 are then abandoned), but their analysis of is tremendously easier in linear perturbation theory. There multiplication of any two inhomogeneous quantities can be neglected<sup>1</sup>, and then three additional steps allow to simplify the analysis further. It is useful to work in the Fourier space, since 1) different  $k$ -modes decouple. Then also 2) different perturbation modes, classified into scalars, vectors and tensor according to their transformation properties under rotations, decouple from each other, and thus satisfy independent evolution equations. In addition, 3) gauge transformations can be used to simplify the analysis. We will here restrict ourselves to linear perturbation theory.

The line-element in the perturbed FRW spacetime can be written as

$$ds^2 = a^2(\tau) \left\{ - (1 + 2\alpha) d\tau^2 - 2(\beta_{,i} + b_i) d\tau dx^i + \left[ g_{ij}^{(3)} + 2 \left( g_{ij}^{(3)} \varphi + \gamma_{|ij} + c_{(i|j)} + h_{ij} \right) \right] dx^i dx^j \right\}. \quad (4.2)$$

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<sup>1</sup>We assume implicitly that inhomogeneities are small, for example  $\rho = \bar{\rho}(t) + \delta\rho(t, \mathbf{x})$ , where  $\delta\rho(t, \mathbf{x}) \ll \rho(t)$ . Thus we approximate  $\delta\rho^2(t, \mathbf{x}) \approx 0$ . Often in perturbation theory (though not in cosmology) one uses an expansion parameter which explicitly multiplies the perturbed quantities.

We characterize the scalar perturbations in a general gauge by the four variables  $\alpha, \beta, \varphi, \gamma$ . Vector perturbations introduce four more degrees of freedom, the divergenceless 3-vector fields  $b_i$  and  $c_i$ . Gravitational waves are described by the two free components of the symmetric, transverse and traceless 3-tensor  $h_{ij}$ . We have thus decomposed the ten independent components of the symmetric  $\delta g_{\mu\nu}$  into three types of perturbations according to their transformation properties under spatial rotations. The vertical bar indicates a covariant derivative based on the Levi-Civita connection of the comoving spatial background metric  $g_{ij}^{(3)}$ . This metric is used to lower and raise spatial indices  $i, j, k \dots$  of the perturbation variables.

The components of the energy-momentum tensor for an imperfect fluid are

$$T_0^0 = -(\bar{\rho} + \delta\rho), \quad T_i^0 = -(\bar{\rho} + \bar{p}) \left( v_{,i} + v_i^{(v)} \right), \quad T_j^i = (\bar{p} + \delta p) \delta_j^i + \Pi_j^i. \quad (4.3)$$

Here  $\rho$  and  $p$  are energy density and pressure, and  $v, v^{(v)}$  are the scalar and vector velocity perturbations<sup>2</sup>, respectively. Background quantities are denoted with an overbar, which we will usually omit when unnecessary. The isotropy of background does not allow anisotropic stress except as a perturbation. This we decompose into the scalar, vector and tensor contributions as

$$\Pi_{ij} \equiv \left( \Pi_{|ij}^{(s)} + \frac{1}{3} \Delta \Pi^{(s)} \right) + \Pi_{(i|j)}^{(v)} + \Pi_{ij}^{(t)}, \quad (4.4)$$

where  $\Delta$  stands for the three-space Laplacian based on the Levi-Civita connection of  $g_{ij}^{(3)}$ . The vector  $\Pi_i^{(v)}$  is divergenceless and the tensor  $\Pi_{ij}^{(t)}$  is symmetric, transverse, and traceless. The four scalar degrees of freedom for the fluid perturbation are therefore  $\delta\rho, \delta p, v$ , and  $\Pi^{(s)}$ , independent components of the divergenceless vectors  $v_i^{(v)}$  and  $\Pi_i^{(v)}$  sum up to four and the tensor describing gravitational waves  $\Pi_{ij}^{(t)}$  has two independent components. Note that the numbers of degrees of freedom agree in the gravitational and in the matter sectors.

Some of these degrees of freedom are due to arbitrariness in separating the background from the perturbations. Consider the transformation of coordinates (2.38) with the  $\epsilon$  absorbed in the vector  $\xi$  and the transformed variables denoted with tildes,

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu, \quad (4.5)$$

Then the relation of the transformed metric at the transformed coordinates,  $\tilde{g}_{\mu\nu}(\tilde{x})$ , to our original  $g_{\mu\nu}(x)$  is given by Eq.(2.41),

$$\tilde{g}^{\mu\nu}(\tilde{x}) = g^{\mu\nu}(x) - \nabla^{(\mu} \xi^{\nu)}, \quad (4.6)$$

However, in cosmological perturbation theory we do not want to identify the metrics at the same point on the manifold, but rather with the same values for the different coordinates, because there is no unique matching of points between the background and the perturbed manifolds. We assume there exists the FRW background. When we want to identify the perturbed part of the metric, we subtract the background from the total, and wish the

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<sup>2</sup>Note that we use the covariant velocity perturbations, sometimes denoted as[114]  $v \equiv a(V - \beta)$ , and  $v_i^{(v)} \equiv a(V_i^{(v)} - b_i)$ .

background to be given by a unique value of the conformal time which does not depend on the gauge. A given value for  $\tau$  then corresponds to slightly different points on the manifold, depending on the coordinate system. In this way the gauge choice determines what is the background in the total spacetime, but the difference between gauges is always of the first order in perturbations. So the transformed metric is defined by

$$\tilde{g}_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x - \xi) = \tilde{g}_{\mu\nu} - g_{\mu\nu,\alpha}\xi^\alpha = g_{\mu\nu}(x) - g_{\mu\nu,\alpha}\xi^\alpha - g_{\mu\alpha}\xi_{,\nu}^\alpha - g_{\nu\alpha}\xi_{,\mu}^\alpha. \quad (4.7)$$

The perturbations in the transformed metric are then given by simply inserting the  $g_{\mu\nu}$  from Eq.(4.2) to the right hand side and equating with the Eq.(4.2) with tildes all over. As an example, consider the 0 – 0 -component:

$$-a^2(1 + 2\tilde{\alpha}) = -a^2(1 + 2\alpha) + 2Ha^2\xi^0 + 2a^2\dot{\xi}^0, \quad (4.8)$$

where we have used the facts that  $\xi$ ,  $\alpha$  and  $g_{i0}$  are perturbations. We can then identify  $\tilde{\alpha} = \alpha - H\xi^0 - \dot{\xi}^0$ . Under a general diffeomorphism  $\xi_i = \xi_{,i} + \xi_i^{(v)}$ , where the first term is the scalar and the second the vector part obeying  $\xi_i^{(v)|i} = 0$ , the transformation rules we find are the following:

$$\tilde{\alpha} = \alpha - H\xi^0 - \dot{\xi}^0, \quad \tilde{\varphi} = \varphi - H\xi^0, \quad \tilde{\beta} = \beta - \xi^0 + \dot{\xi}, \quad \tilde{\gamma} = \gamma - \xi \quad (4.9)$$

for the scalar perturbations and

$$\tilde{b}_i = b_i + a(H\xi_i + \dot{\xi}_i), \quad \tilde{c}_i = c_i - \xi_i^{(v)} \quad (4.10)$$

for the vector quantities. Doing the same for the tensor  $T_{\mu\nu}$  as we did for  $g_{\mu\nu}$ , we can find out how the fluid perturbations behave under gauge transformations:

$$\tilde{\delta} = \delta + 3H(1 + w)\xi^0, \quad \tilde{\delta p} = \delta p - \dot{p}\xi^0, \quad \tilde{v} = v - \xi^0. \quad (4.11)$$

The vector  $v^{(v)}$  is gauge invariant, which means that  $\tilde{v}^{(v)} = v^{(v)}$ .

In the gauge-ready formalism [115, 116, 117, 118] one deals with these gauge degrees of freedom by noting that the homogeneity and isotropy of the background space implies invariance of all physical quantities under purely spatial gauge transformations. Therefore one can trade  $\beta$  and  $\gamma$  to the shear perturbation

$$\chi \equiv a(\beta + \dot{\gamma}). \quad (4.12)$$

where an overdot means derivative with respect to the conformal time  $\eta$ . Since both  $\beta$  and  $\gamma$  vary under spatial gauge transformation, they appear only through the spatially invariant linear combination  $\chi$  in all relevant equations. In addition, one can define the perturbed expansion scalar

$$\kappa \equiv \frac{3}{a}(H\alpha - \dot{\varphi}) - \frac{\Delta}{a^2}\chi, \quad (4.13)$$

To complete the metric transformation laws (4.9), we have

$$\tilde{\chi} = \chi - a\xi^0, \quad a\tilde{\kappa} = a\kappa + [3(\dot{H} - H^2) + \Delta]\xi^0. \quad (4.14)$$

The variable  $\kappa$  is a convenient linear combination, the use of which simplifies some equations, but it is not linearly independent of other perturbations. Only three of the variables  $\alpha$ ,  $\varphi$ ,  $\chi$  and  $\kappa$  are independent. The advantage of using this set of variables is based on the fact that they are spatially gauge-invariant. Writing equations in terms of them, one can conveniently fix the temporal gauge by just setting one of these metric perturbations to zero.

The synchronous gauge, corresponding to  $\alpha = 0$ , is an exception where the gauge mode is removed only up to a constant. The line element in the synchronous gauge can be written as

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij}^S)dx^i dx^j]. \quad (4.15)$$

Conventionally the scalar modes are then defined in the Fourier space by the decomposition

$$h_{ij}^S(\mathbf{x}, \tau) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} [\hat{k}_i \hat{k}_j h(\mathbf{k}, \tau) + 6(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij})\eta(\mathbf{k}, \tau)], \quad (4.16)$$

when  $\mathbf{k} = k\hat{k}$ . This notation corresponds in the gauge-ready notation to setting  $\kappa = -\dot{h}/2a$  and  $\varphi = -\eta$  with  $\alpha = 0$ . Since in this gauge no velocity perturbation is generated to pressureless matter if it does not initially have such, it is convenient to fix the integration constant of the gauge mode by shifting the time slicing such that the cold dark matter velocity perturbation is zero. Therefore synchronous gauge is the frame comoving with CDM.

The Newtonian gauge, also called zero-shear or longitudinal for the reason that there  $\chi = 0$ , is another gauge choice extensively employed in the literature. In this thesis we will write the line element in the Newtonian gauge as<sup>3</sup>

$$ds^2 = a^2(\tau)[-(1 + 2\Psi)d\tau^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j]. \quad (4.17)$$

Then we have  $\varphi = -\Phi$  and  $\alpha = \Psi$ , implying  $\kappa = 3H\Psi$ . Yet one more popular gauge is the comoving one, where, instead of setting any of the metric perturbations to zero, one sets the fluid velocity perturbation  $v$  in Eq.(4.3) to  $v = 0$ . Suitable linear combinations of the above gauge conditions can be considered also. Then any of the metric or fluid quantity is not necessarily set to zero; such gauge conditions however are seldom employed.

Let us briefly consider also the vector (sometimes called rotational) perturbations. Similarly as with the scalar modes, one may exploit the spatially gauge-invariant variable

$$\Psi_i \equiv b_i + \dot{c}_i \quad (4.18)$$

to characterize vector perturbations of the metric. The equations governing the evolution of rotational perturbations are

$$\frac{k^2 - 2K}{2a^2}\Psi_i = (\rho + p)v_i^{(v)}, \quad (4.19)$$

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<sup>3</sup>Our  $\Psi$  ( $\Phi$ ) is the  $\Psi$  ( $-\Phi$ ) of Kodama and Sasaki[119], which is  $\psi$  ( $\phi$ ) of Ma and Bertchinger[120] and  $\Phi_A$  ( $-\Phi_H$ ) of Bardeen[115]. Unfortunately no universal convention exists. Many authors (for examples from standard references, Mukhanov, Feldman and Brandenberger in their review article [121] and Liddle and Lyth in their book [122]) have interchanged  $\phi$  and  $\psi$ .

$$\frac{1}{a^2} \left[ a^4 (\rho + p) v_i^{(v)} \right]^\bullet = -\frac{k^2 - 2K}{2k^2} \Pi_i^{(v)}. \quad (4.20)$$

The first is the field equation (the  $G_i^0$  component), the second one the conservation equation. For example, one can see that the angular momentum of a perfect fluid  $\sim a^4 (\rho + p) v_i^{(v)}$  is conserved. Vector perturbations, even if generated at an early stage, tend to decay in an expanding universe. They are not relevant in most of the models discussed in this thesis.

We will not be much concerned with the tensor perturbations,  $h_{ij}$ , either. They are gauge-invariant by construction, which is easy to check from Eq.(4.6). The tensorial perturbations of the metric describe gravitational waves. Indeed, their evolution is governed by the wave equation

$$\ddot{h}_j^i + 2H\dot{h}_j^i + (k^2 + 2K) h_j^i = a^2 \Pi_j^{i(t)}. \quad (4.21)$$

For a flat universe, an integral solution exists for the superhorizon scales where the gradients and anisotropic stresses can be neglected,

$$h_j^i(\eta, k) = A_j^i - B_j^i \int^\eta \frac{1}{a^2} d\eta. \quad (4.22)$$

Here  $A_j^i$  and  $B_j^i$  are constants for each  $k$ -mode. Tensor perturbations are generated in some inflation models, and in principle a background of gravitational waves could be detected at large scales. However, sensitivity of present detectors is too weak to access the relevant energy scales. In the following we will concentrate on the scalar perturbations, since they are responsible for the formation of cosmological structure.

## 4.2 Scalar perturbations

We begin by listing field equations that govern the evolution of scalar perturbations in a general gauge. The derivation of these is straightforward, and intermediate steps are presented in appendix B. The energy constraint ( $G_0^0$  component of the field equation) is

$$2Ha\kappa + (6K - 2k^2)\varphi = \frac{1}{M^2} a^2 \delta T_0^0 \quad (4.23)$$

and the momentum constraint ( $G_i^0$  component) is

$$a\kappa - (k^2 - 3K) \frac{1}{a} \chi = -\frac{3}{2M^2} \frac{a^2}{k^2} \delta T_i^0. \quad (4.24)$$

The shear propagation equation ( $G_j^i - \frac{1}{3}\delta_j^i G_k^k$  component) reads

$$\frac{1}{a} \dot{\chi} + H \frac{1}{a} \chi - \alpha - \varphi = \frac{3}{2M^2} \frac{a^2}{k^2} \frac{k^j k_i}{k^2 - 3K} \left( \delta T_j^i - \delta_j^i \frac{1}{3} \delta T_k^k \right). \quad (4.25)$$

The Raychaudhuri equation ( $G_k^k - G_0^0$  component) is now given by

$$2a\dot{\kappa} + 4Ha\kappa + \left[ 6(\dot{H} - H^2) - 2k^2 \right] \alpha = \frac{1}{M^2} a^2 \left( -\delta T_k^k - T_0^0 \right). \quad (4.26)$$

These equations generalize the Friedmann equations to a perturbed FRW universe. The conservation equations for the (minimally coupled) fluid with energy-momentum tensor  $T_{\mu\nu}$  give two equations for the scalar perturbations,

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) = -(\rho + p)(kv - \kappa) + \dot{\rho}\alpha, \quad (4.27)$$

which we call the continuity equation (when discussing scalar perturbations), and

$$\frac{1}{a^4(\rho + p)k} [a^4(\rho + p)v]^\bullet = \alpha + \frac{1}{\rho + p} \left( \delta p - \frac{2}{3} \frac{k^2 - 3K}{k^2} \pi^{(s)} \right), \quad (4.28)$$

which can be called the Euler equation. They hold separately for each minimally coupled species of fluid, and for the total matter content even if individual fluids are coupled with each other.

It is useful to define a gauge-invariant perturbation variable  $\mathcal{R}$ , which equals the curvature perturbation of spatial hypersurfaces in the comoving gauge. Without fixing the gauge, this variable can be written in the following way:

$$\mathcal{R} \equiv \varphi + \frac{2}{3(1+w)} \left( \frac{\dot{\varphi}}{H} - \alpha \right), \quad (4.29)$$

where  $w$  is the total EoS of the cosmic fluid. On the other hand, the gauge-invariant total entropy perturbation in the cosmic fluid can be defined as

$$\mathcal{S} \equiv H \left( \frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}} \right). \quad (4.30)$$

This  $\mathcal{S}$  is nonzero only when the pressure perturbation is not determined from the density perturbation alone but depends on more degrees of freedom. By using the transformation laws for the metric, Eq.(4.9), and for the fluid, Eq.(4.11), it is easy to check that both the  $\mathcal{R}$  and  $\mathcal{S}$  are independent of the gauge. Differentiating Eq.(4.29) and using the field equations listed in the previous paragraph, one can arrive at an evolution equation for  $\mathcal{R}$  of the form

$$\frac{3}{2}(1+w)\dot{\mathcal{R}} = - \left( \frac{k}{H} \right)^2 \left[ c_S^2 \left( \varphi - H \frac{1}{a} \chi \right) + \frac{1}{3} \left( \varphi - \alpha + \frac{1}{a} (\dot{\chi} - H\chi) \right) \right] + \frac{9}{2} c_S^2 (1+w) \mathcal{S}, \quad (4.31)$$

where  $c_S^2 \equiv \dot{p}/\dot{\rho}$ . The equation tells us that for adiabatic perturbations ( $\mathcal{S} = 0$ ) the curvature perturbation is constant at superhorizon scales ( $k \ll H$ ).

This is the reason why it is convenient to relate the normalization of perturbations to the initial value of  $\mathcal{R}$  (as we will do in the following). The initial conditions, set presumably at inflation, can be then given deep in the radiation dominated era (at say, near the time of nucleosynthesis), so early that the relevant scales are still all outside the horizon. Analytic solution can be found for perturbations, with the assumption  $w = 1/3$  and the approximation  $k \ll H$ . However, one must specify how the perturbations in individual fluids are related to each other. The most standard assumption is that the primordial perturbations are adiabatic. This is the prediction of the simplest inflation models, and

there is no observational indication for more general initial conditions[123]. Adiabaticity of perturbations[122] means that given any two regions  $A$  and  $B$  in a perturbed universe at an instant  $\tau$ ,  $B$  is identical to  $A$  at some other instant  $\tau + \delta\tau$ . Evolution of the regions is the same, but slightly asynchronous. Then the ratio of fluctuation and time derivative of say a density field is common to all density fields. Thus we can define the entropy between species  $i$  and  $j$  as

$$\mathcal{S}_{ij} \equiv 3H \frac{\delta\rho_i}{\dot{\rho}_i} - \frac{\delta\rho_j}{\dot{\rho}_j} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j}. \quad (4.32)$$

The fractional density perturbation  $\delta$  is defined as  $\delta \equiv \delta\rho/\rho$ . The general adiabatic condition then dictates that all  $\mathcal{S}_{ij}$ 's together with their derivatives vanish. Using the conservation equations (4.27) and (4.28) one finds that  $\dot{\mathcal{S}}_{ij} = 0$  implies  $(k/H)^2(v_i - v_j) = 0$  and thus the entropy perturbation can stay constant at superhorizon scales even when there's difference in the velocity potentials. We have generalized the adiabatic condition for interacting models [2] and for extra anisotropic stresses[5]. Possible deviations from the adiabaticity can be then characterized by  $\mathcal{S}_{ij}$ 's for different pairs of fluids. Conventionally, these are expressed in terms of the relation of photon perturbations and those in each other fluid present. Entropic initial conditions are called isocurvature perturbations. There are five different modes: the adiabatic one plus baryon, dark matter and neutrino isocurvature mode. In addition, there exists a neutrino velocity isocurvature mode, satisfying  $\delta_\nu = \delta_\gamma$ , but not  $v_\nu = v_\gamma$  [124]. A baryon and CDM isocurvature modes are in practice observationally indistinguishable, up to a normalization depending on  $\Omega_b$  and  $\Omega_c$ . The neutrino isocurvature mode would diverge in the Newtonian gauge as  $a \rightarrow 0$ , but it can be regarded as a possibly physical mode in the synchronous gauge. If one excludes the modes for which the curvature perturbation would be diverging as  $a \rightarrow 0$ , one finds that only four independent dominant modes<sup>4</sup> are left in the Newtonian gauge[125, 126]. A dark energy component introduces another isocurvature mode [127] (such a case is to be discussed later in the thesis). However, again excluding diverging modes, it has been claimed that still only four independent modes appear[125] when the dark energy is modelled as tracking quintessence. Thus it seems that in such a case the dark energy isocurvature mode is not an independent degree of freedom but any initial conditions with arbitrary dark energy perturbations can be expressed as a linear combination of the adiabatic and the three basic isocurvature modes. Thus, in the (not most general) case that we do not include the dark energy and the neutrino velocity isocurvature mode, a four-component vector should be specified in order to fix the initial conditions for cosmological perturbations:  $(\mathcal{R}, \mathcal{S}_{b\gamma}, \mathcal{S}_{c\gamma}, \mathcal{S}_{\nu\gamma})$ . The adiabatic mode (also called isentropic) is then  $(\mathcal{R}, 0, 0, 0)$ , and the pure baryon isocurvature case would be  $(0, \mathcal{S}_{b\gamma}, 0, 0)$ , etc. Thus the name isocurvature: for these modes the comoving curvature stays unperturbed.

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<sup>4</sup>This seems to suggest that the number of non-diverging modes depends on gauge.



## 4.3 CMB Physics

### 4.3.1 The multipole expansion

The observed temperature anisotropy today in direction  $\hat{\mathbf{n}}$  is conveniently expanded in terms of spherical harmonics:

$$\Theta(\tau_0, \hat{\mathbf{n}}) \equiv \frac{\delta T}{T}(\tau_0, \hat{\mathbf{n}}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}), \quad (4.33)$$

where the sums are over  $\ell = 1, 2, \dots, \infty$  and  $m = -\ell, \dots, \ell$ . The observed angular temperature power spectrum is then defined as the average of the observed variances

$$\hat{C}_\ell \equiv \frac{1}{2\ell + 1} \sum_m a_{\ell m}^* a_{\ell m}. \quad (4.34)$$

However, we do cannot predict any particular  $a_{\ell m}$ , but rather the distribution from which they were drawn. The variance of  $a_{\ell m}$  is independent of  $m$ . Theoretically, we can predict only this variance, but not its actual realization. The expected CMB spectrum is the expectation value of the theoretical spectra,

$$C_\ell \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell + 1} \sum_m \langle a_{\ell m}^* a_{\ell m} \rangle. \quad (4.35)$$

These equalities hold because the primordial fluctuations, and thus the CMB anisotropy, is assumed to be result of a statistically isotropic random process. The different  $a_{\ell m}$  are independent random variables, and thus we have that

$$\langle a_{\ell m}^* a_{\ell' m'} \rangle = C_\ell \delta_{\ell \ell'} \delta_{m m'}. \quad (4.36)$$

The so called cosmic variance arises from the fact that we can measure only a finite number of realizations of these random variables. For the quadropole, we have  $2 \times 2 + 1 = 5$  of them, which leaves significant uncertainty. One sees that the uncertainty due to cosmic variance is  $\sqrt{2/(2\ell + 1)}$  for each multipole  $\ell$ . So when considering smaller scales, say  $\ell$  equal to few dozens, the cosmic variance is smaller than the uncertainty due to measurement errors.

We want to have the  $C_\ell$  spectrum determined by the spectrum of primordial curvature perturbation  $\mathcal{R}$ ,

$$\mathcal{P}_\mathcal{R}(k) \equiv \frac{k^3}{2\pi^2} \langle \mathcal{R}_\mathbf{k}^* \mathcal{R}_\mathbf{k} \rangle = A k^{n_S - 1}, \quad (4.37)$$

where  $A$  is the amplitude and  $n_S$  the (scalar) spectral index of the primordial fluctuation spectrum. We define the transfer function  $T_\Theta(k, \mu)$  so that in Fourier space  $\Theta(\tau_0, \mathbf{k}, \mathbf{n}) = T_\Theta(k, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) \mathcal{R}_\mathbf{k}$ . Next we expand the transfer function in Legendre series and Fourier transform the temperature anisotropy to position space. Then we have that

$$\Theta(\tau_0, \mathbf{x}, \mathbf{n}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_\ell (-i)^\ell (2\ell + 1) \int T_\Theta(k, \ell) P_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) \mathcal{R}_\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} d^3k. \quad (4.38)$$

By using equations (4.33)-(4.37), and some useful formulas for the exponent function and spherical harmonics, we arrive at the following expression for the angular power spectrum:

$$C_\ell = 4\pi \int \frac{dk}{k} T_\Theta^2(k, \ell) \mathcal{P}_\mathcal{R}(k). \quad (4.39)$$

It remains to find the transfer function.

We begin from the Boltzmann equation for photons [10], with source term from Thomson scattering included:

$$\dot{\Theta} + ik\mu\Theta - \dot{\kappa}_T\Theta = \dot{\Psi} - i\mu k\Phi - \dot{\kappa}_T\left(\frac{1}{4}\delta_\gamma + i\mu kv_b - \frac{1}{2}P_2(\mu)\Pi\right). \quad (4.40)$$

The scattering term depends on the number density of free electrons  $n_e$ ,

$$\dot{\kappa}_T \equiv -an_e\sigma_T, \quad (4.41)$$

where  $\sigma_T$  is the Thomson cross section. Here again  $\Theta$  is a function of conformal time,  $k$  and  $\mu$ , the cosine of the angle between  $\mathbf{k}$  and  $\hat{\mathbf{n}}$ . The LHS of Eq.(4.40) can be written as

$$e^{-i\mu k\tau}e^{+\kappa_T(\tau)}\frac{d}{d\tau}[e^{i\mu k\tau}e^{-\kappa_T(\tau)}\Theta]. \quad (4.42)$$

Thus the derivative of the square brackets above is the RHS of equation (4.40) multiplied by the two exponents inside those brackets. So one can integrate that product to get the term inside the square brackets today. When that is again multiplied by the inverse of those exponents, evaluated today, the result is the present temperature anisotropy:

$$\Theta(\mu, \tau_0) = \int_0^{\tau_0} e^{i\mu k(\tau-\tau_0)}e^{\kappa_T(\tau)-\kappa_T(\tau_0)} \left[ \dot{\Psi} - i\mu k\Phi + \dot{\kappa}_T \left( \frac{1}{4}\delta_\gamma + i\mu kv_b - \frac{1}{2}P_2(\mu)\Pi \right) \right] d\tau. \quad (4.43)$$

To get rid of the  $\mu$ -prefactors, we integrate this by parts (twice, since  $P_2(\mu)$  involves  $\mu^2$ ). The boundary terms can be dropped, because at  $\tau = 0$  they vanish, and at  $\tau = \tau_0$  they would contribute only to the monopole and the dipole. The result is conveniently expressed in terms of the visibility function,  $g \equiv -\dot{\kappa}_T e^{\kappa_T(\tau)-\kappa_T(\tau_0)}$ :

$$\begin{aligned} \Theta(\mu, \tau_0) = & \int_0^{\tau_0} e^{i\mu k(\tau-\tau_0)} \left[ g \left( \frac{1}{4}\delta_\gamma - \dot{v}_b + \Phi + \frac{3}{4k^2}\ddot{\Pi} + \frac{1}{4}\Pi \right) \right. \\ & \left. - \dot{g} \left( v_b + \frac{3}{4k^2}\dot{\Pi} \right) + \ddot{g} \frac{3}{4k^2}\Pi + e^{-\kappa_T(\tau)-\kappa_T(\tau_0)} \left( \dot{\Phi} + \dot{\Psi} \right) \right] d\tau. \end{aligned} \quad (4.44)$$

We drop  $\kappa_T(\tau_0)$ , since it equals to zero. The exponent function  $e^{ik(\tau-\tau_0)}$  can be expanded in terms of  $j_l$  and  $P_l$ . Remembering the definition of the transfer function, we are then ready to pick the coefficients in its Legendre expansion. They are:

$$\begin{aligned} \mathcal{R}_k T_\Theta(k, l) = & \int_0^{\tau_0} \left[ g \left( \frac{1}{4}\delta_\gamma - \dot{v}_b + \Phi + \frac{3}{4k^2}\ddot{\Pi} + \frac{1}{4}\Pi \right) \right. \\ & \left. - \dot{g} \left( v_b + \frac{3}{4k^2}\dot{\Pi} \right) + \ddot{g} \frac{3}{4k^2}\Pi + e^{-\kappa_T(\tau)-\kappa_T(\tau_0)} \left( \dot{\Phi} + \dot{\Psi} \right) \right] j_l[k(\tau_0 - \tau)] d\tau. \end{aligned} \quad (4.45)$$

The prefactor  $\mathcal{R}$  appears in the left hand side since we want to relate the perturbations to their primordial values: transfer function gives the time evolution of the perturbations for each  $k$ -mode, but their amplitude is determined by the primordial spectrum<sup>5</sup>. The

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<sup>5</sup>RHS of Eq.(4.45) can be considered as the right expression for the transfer function when the initial curvature perturbation is set to unity. Then the normalization and possible tilt in the primordial spectrum are taken into account in Eq.(4.39).

anisotropy sources in (4.45), are multiplied by a Bessel function. That is the geometrical part of the transfer function, which governs how the anisotropies contribute to different multipoles in the spherical expansion. In addition, the perturbations are weighted by the visibility factors,  $g$ ,  $g'$  and  $e^{\kappa_T(\tau)-\kappa_T(\tau_0)}$ , so that the anisotropy is gathered in the integral from the relevant parts of the universe. Let us briefly discuss each of the source terms.

- $\frac{1}{4}\delta_\gamma$  is the temperature anisotropy present at last scattering. Because in thermal equilibrium  $\rho_\gamma$  is proportional to  $T^4$ , there is the factor  $1/4$  in front of the fractional density perturbation. That term is the primary cause of the acoustic peaks at the smaller angular scales.
- The Sachs-Wolfe effect stems from the  $\Phi$ . The photons coming from overdense regions suffer a loss of energy due to their climbing out of the gravitational well induced by the overdensity. A proper temperature anisotropy taking into account the gravitational shift is then  $\delta_\gamma/4 + \Phi$ , and since it is weighted by the  $g$  which is sharply peaked at last scattering, the main contribution comes directly from there.
- Also the combined effect of terms  $g'v_b + gv'_b$ , which corresponds to Doppler shift of photons due to movement of baryons, is also important for the shape of the angular power spectrum<sup>6</sup>.
- $\dot{\Psi} + \dot{\Phi}$  is responsible for the integrated Sachs-Wolfe effect (ISW). The red- and blueshifts of photons travelling into and out from static gravitational wells along their path to us cancels out. This canceling is not exact if the gravitational potentials evolve. During matter domination, the gravitational potentials are constant. Therefore there are two distinct ISW contributions: the early one when radiation is not yet negligible and the late one when dark energy takes over matter. The early ISW comes from the scales corresponding roughly to the last scattering surface, and the late one comes mainly from large scales.
- The terms involving  $\Pi$  and its derivatives have smaller but still non-negligible effects on the CMB spectrum. The higher multipoles of the photon distribution are suppressed by the tight coupling to baryons, but begin to evolve at the decoupling. In solving the evolution of  $\Pi$ , one has to take into account that it is coupled to higher multipoles. But since they are suppressed, it is enough to consider multipoles up to about  $\ell = 8$  to get accurate results. In addition,  $\Pi$  is coupled to the polarization, which it in turn sources. Thus polarization is smaller than the temperature anisotropy, and cannot be detected with equal accuracy. Among many other important aspects of CMB, here we omitted a more detailed study of polarization. Exotic dark energy might couple non-minimally to photons[128], but usually we have only indirect effect from dark energy perturbations to the polarization. Schematically the coupling goes like  $\delta_{de} \leftrightarrow \Psi \leftrightarrow v_\gamma \leftrightarrow \Pi_\gamma \leftrightarrow \Pi_P$ . Nevertheless, at least in future,

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<sup>6</sup>Adiabatic initial conditions require that the  $\delta_\gamma$  is negative (of the opposite sign than  $\mathcal{R}$ ) at super-horizon scales. At decoupling, the monopole contribution to effective temperature,  $\delta_\gamma/4 + \Psi$ , is still smaller than zero at large scales. The first acoustic peak however corresponds to compression of the photon-baryon fluid. Would the anisotropy at  $\tau_{LS}$  be due only to the photon density and the gravitational potential, the spectrum would have a minimum before the rise to the first acoustic peak. The reason that there is no such minimum is the Doppler (and to a smaller amount, the early ISW) effect.

precise constraints on dark energy should also take advantage of the information inferable from the polarization spectra.

### 4.3.2 Recombination

The previous equations involve also the ionization history of the universe. This comes about through the terms including the visibility function  $g$  and the optical depth  $\kappa_T$  (see Eq.(4.41). In this subsection we give a schematic derivation of the Peebles equation for the ionization fraction of electrons, which is not necessary to go through to follow the main development of the rest of thesis: the results of recombination calculation are presented in Fig.(4.1).

The forming of neutral hydrogen occurs via the process  $e^- + p \rightarrow H + \gamma$ . Since this process does not remain in equilibrium during the recombination, we have to consider the Boltzmann equation. In an expanding universe, it can be written as

$$\begin{aligned} \frac{1}{a^3} \frac{dn_e a^3}{dt} &= \int \frac{d^3 p_e}{(2\pi)^3 2E_e} \int \frac{d^3 p_p}{(2\pi)^3 2E_e} \int \frac{d^3 p_H}{(2\pi)^3 2E_e} \int \frac{d^3 p_\gamma}{(2\pi)^3 2E_e} \\ &\times (2\pi)^4 \delta(E_e - E_p - E_H - E_\gamma) \delta^3(p_e - p_p - p_H - p_\gamma) |\mathcal{M}|^2 \\ &\times \{f_H f_\gamma [1 - f_e][1 - f_p] - f_e f_p [1 + f_H][1 + f_\gamma]\}. \end{aligned} \quad (4.46)$$

Here the scattering amplitude  $\mathcal{M}$  is proportional to the fine structure constant. A crucial assumption is now that the scattering takes place so rapidly that all the species involved remain in kinetic equilibrium. Furthermore the recombination happens at temperatures  $\sim 0.1$  eV, enough below the rest mass of electron to use the simple Maxwell-Boltzmann forms for the distribution functions. Then the  $\{\}$ -term in the previous equation, knowing that the chemical potential of photons is zero and that the reaction conserves energy, can be written as

$$e^{-(E_e + E_p)/T} [e^{\mu_H/T} - e^{(\mu_e + \mu_p)/T}] = e^{-(E_e + E_p)/T} \left[ \frac{n_H}{n_{(0)}} - \frac{n_e n_p}{n_{e(0)} n_{p(0)}} \right]. \quad (4.47)$$

The second equality follows from thermodynamics: fugacity  $e^{-\mu/T}$  is equivalent to the ratio of number density  $n_i$  to its equilibrium value  $n_i^{(0)}$ . Thus in chemical equilibrium the RHS of (4.46) vanishes. The condition that the  $[\ ]$ -term in Eq.(4.47) is zero implies the Saha equation,

$$\frac{X_e}{1 - X_e^2} = \frac{1}{\hat{n}_H} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-B_H/T}, \quad (4.48)$$

where  $B_H$  is the Hydrogen binding energy  $B_H = m_p + m_e - m_H \approx 13.6$  eV. Here we have taken into account that since the universe should be electrically neutral, we  $n_e = n_p$ , and defined the free electron fraction as  $X_e \equiv n_e / \hat{n}_H$ , where  $\hat{n}_H$  is the total number density of hydrogen nuclei; if we neglect helium, it is equal to the baryon number density,  $\hat{n}_H = n_B = n_p + n_H$ .

The prefactor of the  $[\ ]$ -term in the of Eq.(4.46) can be absorbed to the definition of thermally averaged cross section, introducing of which will make our starting point (4.46)

look considerably simpler. The thermally averaged cross section is

$$\begin{aligned} \langle \sigma v \rangle &\equiv \frac{1}{n_e^{(0)} n_p^{(0)}} \int \frac{d^3 p_e}{(2\pi)^3 2E_e} \int \frac{d^3 p_p}{(2\pi)^3 2E_e} \int \frac{d^3 p_H}{(2\pi)^3 2E_e} \int \frac{d^3 p_\gamma}{(2\pi)^3 2E_e} \\ &\times e^{(E_e + E_p)/T} (2\pi)^4 \delta(E_e - E_p - E_H - E_\gamma) \delta^3(p_e - p_p - p_H - p_\gamma) |\mathcal{M}|^2. \end{aligned} \quad (4.49)$$

Using then our previous definitions and the result (4.47) in equation (4.46), we have

$$\frac{1}{a^3} \frac{dn_e a^3}{dt} = \hat{n}_H \langle \sigma v \rangle \left\{ (1 - X_e) \frac{n_e^{(0)}}{n_p^{(0)}} n_H^{(0)} - X_e^2 \hat{n}_H \right\}. \quad (4.50)$$

Now we express  $n_e$  in the LHS as  $X_e \hat{n}_H$ , and note that  $\hat{n}_H a^3$  is a constant, so we can divide the  $\hat{n}_H$  from both sides of the equation. Naming the coefficients conventionally, we have the Peebles equation:

$$\frac{dX_e}{dt} = (1 - X_e)\beta - X_e^2 \hat{n}_H \alpha^{(2)}. \quad (4.51)$$

The terms in the equation are

- $\alpha^{(2)} \equiv \langle \sigma v \rangle$  is the recombination rate. It has the superscript <sup>(2)</sup> (in addition to not to confuse with the fine structure constant) because the recombination occurs via electron capture to the excited states of the hydrogen. Direct electron captures to the ground state produce a photon which immediately ionizes another neutral atom; thus the net effect is zero. The recombination rate is approximated by

$$\alpha^{(2)} = 9.780274 \frac{\alpha^2}{m_e^2} \sqrt{\frac{R_H}{T}} \ln \frac{B_H}{T},$$

$\alpha$  being the fine structure constant and  $R_H$  the Rydberg constant appearing in the formulas for spectral lines emitted by atomic hydrogen,  $R_H = m_e e^4 / [(4\pi B_H)^2 \hbar^3 4\pi c]$ .

- $\beta \equiv \alpha^{(2)} \left(\frac{m_e T}{2\pi}\right)^{\frac{3}{2}} e^{-B_H/T}$  is the ionization rate. It is equal to the recombination rate multiplied by the factor which is calculated using the well-known equilibrium distributions.

To arrive at sufficient accuracy, one should modify Eq.(4.51) by the following correction:

$$\frac{dX_e}{dt} = C_r \left[ (1 - X_e)\beta - X_e^2 \hat{n}_H \alpha^{(2)} \right], \quad (4.52)$$

where

- $C_r$  is a reduction factor, which is included to account for the decays from the excited states. A capture of an electron to  $n = 2$  state results in net recombination only if the expansion redshifts the Lyman alpha photon (emitted during the transition to the ground state) enough that it cannot ionize another atom, or if the transition  $2s \rightarrow 1s$  occurs via release of two less energetic photons. The rate for the two-photon

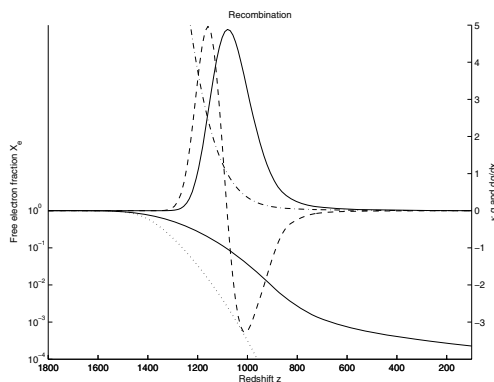


Figure 4.1: On the bottom there are the Peebles (solid line) and Saha (dotted line) solutions to the fraction of free electrons on logarithmic scale as a functions of redshift. On the top there are the corresponding optical depth  $\int_{\tau_z}^{\tau_0} \kappa_T d\tau$  (dash-dotted line), the visibility function  $g$  (solid line) and the derivative of the visibility function  $dg/d \log(a)$  (dashed line). The visibility function is in units of  $H_0$ , and the its derivative is in units of  $10H_0$  (so that it fits into the same picture). One sees that the equilibrium solution (Saha) is not a good approximation to the free electron fraction (from the Peebles equation). The former is much shallower, and does not drop to exactly zero, since some electrons remain free in the expanding universe. The visibility function, which picks up anisotropies in the transfer function integral 4.45, is very sharply peaked at last scattering. The optical depth rises rapidly there, meaning it is impossible to see further.

decay is  $\Lambda_{2s \leftarrow 1s} = 8.227 \text{sec}^{-1}$ , the Lyman alpha photons are produced by the rate  $\beta^{(2)} = \beta e^{3B_H/4T}$  and redshifted by the rate

$$\Lambda_\alpha = H \frac{(3B_H)^3}{(8\pi)^2 n_{1s}},$$

where the number density of electrons in the 1s orbit is approximated by  $(1 - X_e)\hat{n}_H$ . The reduction factor is the ratio of rates to the recombining channels to the the sum of all the rates, i.e.

$$C_r = \frac{\Lambda_{2s \leftarrow 1s} + \Lambda_\alpha}{\Lambda_{2s \leftarrow 1s} + \Lambda_\alpha + \beta^{(2)}}.$$

We define the optical depth to redshift  $z$  as the integral of  $\kappa_T$  from  $\tau(z)$  to  $\tau_0$ . The visibility function we had defined as  $g(\tau) \equiv -e^{-\kappa_T(\tau)} \dot{\kappa}_T(\tau)$ . We calculate the optical depth by numerical integration from our solution for the free electron fraction. Then we can immediately calculate the visibility function. The derivative of the visibility function is computed by numerical differentiation. We plot these functions in figure (4.1). We have checked that as implied by its definition, the integral of the visibility function equals unity,  $\int g d\tau = 1$ .

## Chapter 5

# Parameterizations

In this section we will discuss perturbations in dark energy in terms of general parameterizations. Without specifying the model, we assume that the background behaviour of dark energy is given in terms of a modified Friedmann equation (subsection 5.1) or as a fluid with negative pressure (subsection 5.2). We then discuss various possibilities to deal with perturbations.

As a warm-up we look at distribution of a negative-pressure fluid in a couple of simple background spacetimes other than the FRW universe. We treat the medium as a "test fluid", which does not backreact to the metric. Then the fluid obeys its continuity equation, but we neglect the contribution of the fluid in the field equation and thus assume the metric is the vacuum solution. This is not perhaps a good approximation but allows easy calculations (for similar considerations, see Ref.[129]). A reader not curious of behaviours of "test fluids" should jump to the next subsection.

First of the examples is an anti de Sitter spacetime (AdS). Such is described by the metric

$$ds^2 = dy^2 + e^{2y/l} (-dt^2 + dx^2 + dz^2). \quad (5.1)$$

This is the maximally symmetric vacuum solution to Einstein equations with a negative cosmological constant. AdS is negatively curved with  $R = -12/l^2$ , the curvature radius depending on the length scale  $l > 0$ . Each slice of constant  $y$  here is however a 1+2 dimensional Minkowski space, and as one goes far along the  $y$ -axis, the space begins to look flat at  $y \rightarrow \infty$ . Consider a matter with energy density  $\rho(y)$  and pressure  $p(y)$  in form of a perfect fluid Eq.(2.7). Note that for simplicity we now assume the distribution depends only on the coordinate  $y$ . The energy-momentum conservation (2.10) gives now

$$\nabla_\mu T^\mu_y = 0 \Rightarrow \frac{dp(y)}{dy} + \frac{1}{l} (p(y) + \rho(y)) = 0. \quad (5.2)$$

Assuming a constant EoS  $w$ , an explicit solution is simply

$$\rho(y) = \rho(0) \exp \left[ -\frac{y}{l} \left( 1 + \frac{1}{w} \right) \right], \quad (5.3)$$

where  $\rho(0)$  is a constant. We can note the following.

- For usual matter, with  $w > 0$ , the density becomes large in the more curved region of the space, where  $y$  is large and negative. In particular, for dust (which is nearly pressureless, but with small positive pressure), Eq.(5.3) indicates a gravitational instability: collapse occurs.
- Quintessential matter, with its pressure given by  $-1 < w < 0$ , behaves in an opposite way. The distribution tends to get thicker towards the  $y \rightarrow \infty$  boundary. The limit  $w = -1$  corresponds to the cosmological constant: the density is then uniform.
- Phantom matter, with  $w < -1$ , would be distributed in same manner as ordinary matter, though a bit more smoothly.

We will shortly see that inhomogeneities in matter depend somewhat similarly on its EoS in cosmological spacetimes with more realistic treatment. Then also we have that closer  $w$  is to zero, the more efficiently matter collapses and that closer  $w$  is to  $-1$ , the more uniformly the matter is distributed.

Let us however consider also a Schwarzschild-like metric (mentioned in section 2),

$$ds^2 = -e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2 d\Omega_{(2)}^2. \quad (5.4)$$

where the  $d\Omega_{(2)}^2$  is the metric of a two-dimensional sphere with unit radius, and obviously  $r$  is the radial coordinate. We naturally assume a spherically symmetric distribution of matter,  $\rho = \rho(r)$ ,  $p = p(r)$ . This time the energy-momentum conservation (2.10) gives

$$\nabla_\mu T_r^\mu = 0 \Rightarrow \frac{dp(r)}{dr} + \frac{df(r)}{dr} (p(r) + \rho(r)) = 0. \quad (5.5)$$

Assuming again a constant EoS  $w$ , this is easy to solve. One obtains

$$\rho(r) = \rho_0 \exp \left[ - \left( 1 + \frac{1}{w} \right) f(r) \right], \quad (5.6)$$

where the boundary condition is that  $\rho = \rho_0$  when  $f = 0$ . For concreteness, we consider the Schwarzschild metric that is given by

$$f(r) = \frac{1}{2} \log \left( 1 - \frac{r_0}{r} \right), \quad (5.7)$$

where  $r_0$  is the Schwarzschild radius (for an object of mass  $m$ ,  $r_0 = 2Gm/c^2$ ). Then the solution (5.6) becomes

$$\rho(r) = \rho_0 \left( 1 - \frac{r_0}{r} \right)^{\frac{1}{2}(1+1/w)}. \quad (5.8)$$

So the constant  $\rho_0$  is the density far away at large  $r$ . The shape of the distribution as a function of the radius  $r > r_0$  depends on the EoS in analogy with our list in the AdS case.

- When  $w > 0$ ,  $\rho$  decreases as a function of  $r$ , in other words matter is localized near the horizon. Specifically, dust is again instable.
- If  $-1 < w < 0$ ,  $\rho$  is an increasing function of  $r$ , and thus this kind of matter delocalizes. However, if  $w = -1$ , the distribution is uniform again.



- Phantom matter, with  $w < -1$ , tends to clump near the horizon.

Thus it seems that dust-like matter gathers near a star while quintessence-like energy becomes less dense as a gravitational well is approached. On the other hand, the above results seem to tell us that density of phantom-like energy becomes larger near a star. These conclusions agree roughly with our general picture where cold dark matter forms structures but dark energy should be rather uniformly distributed. However, we can gain a more detailed view within cosmological perturbation theory, where we will also learn that some of the above conclusion are not quantitatively accurate. Namely, we find that actually the phantom and quintessence-like energy components switch their roles in the sense that it is dark energy fluid characterized by  $w < -1$  which is driven away from overdensities; inhomogeneities in a fluid obeying equation of state parameterized by  $-1 < w < 1$  falls into gravitational well, just not as eagerly as usual matter. Formulawise, in cosmological perturbation equations  $1 + w$  determines the difference, whereas in the previous examples it was  $1 + 1/w$ . The culprit for the difference is our test-fluid approximation, which neglects the gravitational effects of the medium itself.

## 5.1 Modified Friedmann equations

In Table 3.2 we listed some examples of effective energy densities in unified models of dark matter and dark energy. It is possible to consider such models as cosmologies described by a modified Friedmann equation. With the natural assumption that particle number is conserved in an expanding universe,  $\rho_c \sim a^{-3}$ , we can ascribe to each of the models in Table 3.2 a Friedmann equation which generalizes the usual  $3H^2 = a^2 \rho_c$  to  $3H^2 = a^2 \rho_K(\rho_c)$ . To be concrete, the function  $\rho_K(\rho)$  reads in the Modified polytropic Cardassian (MPC) case

$$\rho_K = \rho_c [1 + B \rho_c^{-q\nu}]^{1/q} \quad (5.9)$$

and in the leaking gravity case (DGP model)

$$\rho_K = \rho_c + \frac{3M^2}{2r_0^2} \mp \sqrt{\frac{9M^4}{4r_0^4} + \frac{3M^2}{r_0^2} \rho_c}, \quad (5.10)$$

where we have replaced the parameters  $A$  and  $B$  in the latter case with  $\rho_c$  and  $r_0$ , which can be interpreted as the physical CDM density and the cross-over scale that determines when the extra-dimensional effects become important in this model (though it is not always simply  $r_0$ ). Explicitly,  $\rho_c^{(0)} = A$  and  $r_0^2 = 3M^2/(2B)$ . Equation (5.10) can be derived from a higher dimensional set-up taking into account quantum fluctuations[18, 20]. However, most of the modified Friedmann equations that have been invoked in the literature are phenomenological attempts to describe the cosmic acceleration without appeal to a dark energy component, and lack a consistent derivation from more fundamental grounds.

Nevertheless, one would expect that such modifications bring with them other observable consequences besides the acceleration of the background expansion. As we have seen, cosmological perturbation theory involves more physics than just the evolution of the background scale factor. Therefore the evolution of perturbations cannot be determined given just a form of the modified Friedmann equation. To derive perturbation equations

in each of the cases, one should know the covariant action from which the supposed Friedmann equation would ensue. However, such has not been given for most of these models. Still, it is possible to find some general conditions, which, if satisfied, would determine the perturbation theory completely. Given a Friedmann equation, like those in Table 3.2, one can read the effective energy density (which we will call  $\rho_K$  in the following). Were the modifications due to exotica in the gravitational sector or in the matter sector (or in the both, for that matter), one knows that they can be written in the form of an effective energy-momentum tensor with its properties at the background level determined by the  $\rho_K$ . A natural first step towards the linear order is to write down the covariant conservation of the energy-momentum, since it, as discussed in section 2, can be trusted in very general circumstances. Thus, in the obvious notation for effective energy momentum tensor, we have

$$\nabla_\mu T_K^{\mu\nu} = 0. \quad (5.11)$$

In the background this yields  $\rho_K + 3H(1+w_K)\rho_K = 0$ , which in fact was exploited before to find Eq.(3.12). However, we should somehow take into account that the density  $\rho_K$  involves also the dark matter density  $\rho_c$ . Let us denote  $\rho_X = \rho_K - \rho_c$ . To the background order, all of these components are separately conserved, which follows from the assumptions mentioned this far. When perturbations are considered, two basic possibilities arise. Either  $\rho_X$  and  $\rho_c$  are both conserved (case III), or then they are both not conserved (case II).

### 5.1.1 Case II: assuming interactions

Let us begin with an assumption that cold dark matter is self-interacting in such a way that  $\rho_X$  corresponds to the energy density in fields (which might be interpreted as particles) mediating such hypothetical interaction. Then  $\nabla_\mu T_c^{\mu\nu} = -\nabla_\mu T_X^{\mu\nu} \equiv T^\nu \neq 0$ . The background continuity equations for the component densities are then

$$\dot{\rho}_X + 3H(1+w_X)\rho_X = -T_0 = -(\dot{\rho}_c + 3H\rho_c) \quad (5.12)$$

where the time component of the four-vector  $T_\mu$ ,  $T_0 \neq 0$ , is an energy transfer function characterizing the interaction. Given  $w_X$ , it is straightforward to find the two densities and the energy transfer  $T_0$  as functions of the scale factor. For the MPC case, we would get

$$\rho_c = \frac{(1 + \frac{1}{w_X})A + Ba^{-3(1+\alpha)}}{[A + Ba^{-3(1+\alpha)}]^{\frac{\alpha}{1+\alpha}}}, \quad \rho_X = \frac{-A}{[A + Ba^{-3(1+\alpha)}]^{\frac{\alpha}{1+\alpha}}}. \quad (5.13)$$

$$T_0 = \frac{3H[(1+w_X)A + w_X Ba^{3(1+\alpha)}]}{A + Ba^{-3(1+\alpha)}} \rho_X \quad (5.14)$$

It is useful to introduce effective equation of state parameters such that

$$\dot{\rho}_X + 3H(1+\hat{w}_X)\rho_X = 0, \quad \dot{\rho}_c + 3H(1+\hat{w}_c)\rho_c = 0. \quad (5.15)$$

One notices that now  $\hat{w}_X = w_X + T_0/(3H\rho_X)$  and  $\hat{w}_c = -T_0/(3H\rho_c)$ .

Then we can consider the perturbations in this system of interacting fluids. Since we know the densities and pressures for them as unique functions of the scale factor, we can determine the perturbations if we assume these to be adiabatic. Then the pressure perturbations are to be given by the individual adiabatic sound speed,  $c_i^2 \equiv dp_i/d\rho_i$ , which for the  $c$ -component vanishes:

$$\delta p_X = c_X^2 \delta \rho_X, \quad \delta p_c = 0. \quad (5.16)$$

These hold unless we introduce some internal entropy by hand. Similarly, we get the adiabatic values for the fractional density perturbations of these components:

$$\delta_X = \frac{(1 + \hat{w}_X)}{(1 + w)} \delta, \quad \delta_c = \frac{(1 + \hat{w}_c)}{(1 + w)} \delta. \quad (5.17)$$

It is then simple to find that the effective total sound speed squared coincides with the adiabatic one:

$$\frac{\delta p}{\delta \rho} = \frac{\delta p_X}{\delta \rho_X + \delta \rho_c} = \frac{\dot{p}(1 + \hat{w}_X)\rho_X}{\dot{\rho}_X[(1 + \hat{w}_X)\rho_X + (1 + \hat{w}_c)\rho_c]} = \frac{dp}{d\rho}. \quad (5.18)$$

Thus the perturbation (as well as the background) evolution is just the same as if it was dominated by a single fluid with energy density  $\rho_K$  and no entropy perturbation.

In this light the attempts to revive the unified models by decomposing them into interacting fluids seem futile. Such attempts have been undertaken after it was noticed that the unified models described by (5.9), together with its various subcases (see Table 3.2) are incompatible with the CMB and LSS observations, unless both of the parameters  $q$  and  $\nu$  are tuned extremely close to 1, which reproduces the background expansion of the  $\Lambda$ CDM model [130, 1, 131]. To see the reason for this, we have to consider only the late universe, where radiation can be neglected, since earlier  $\rho_K$  is negligible in these models. Then, by using the field equations in the longitudinal gauge, we can find that the metric potential  $\Phi$  evolves according to

$$\ddot{\Phi} + 3(1 + c_K^2)H\dot{\Phi} + 3(c_K^2 - w_K)\Phi = c_K^2 \nabla^2 \Phi - \frac{9}{2}(1 + w)c_K^2 \mathcal{S}_K. \quad (5.19)$$

Here we have only taken into account the unified fluid, but more realistically one should also include baryons into the picture. It would amount to just changing the subindices from  $K$  to  $T$  (for total), but since inclusion of baryons would not affect our qualitative conclusions we omit this complication here. Note that according to our derivations above, the gauge-invariant entropy perturbation

$$\mathcal{S}_K \equiv H\left(\frac{\delta p_K}{\dot{p}_K} - \frac{\delta \rho_K}{\dot{\rho}_K}\right) \quad (5.20)$$

vanishes. It is useful to turn the evolution equation for the gravitational potential into the corresponding one for the comoving gauge density perturbation  $\delta_K$ , by using the Poisson equation

$$\nabla^2 \Phi = 4\pi G a^2 \rho \delta_K. \quad (5.21)$$

The result is

$$\begin{aligned}\ddot{\delta}_K + [2 - 3(2w_K - c_K^2)]H\dot{\delta} &= \frac{3}{2}(1 - 6c_K^2 + 8w_K - 3w_K^2)H^2\delta \\ &= -\frac{k^2}{a^2}[\delta - 3(1 + w_K)\mathcal{S}]c_K^2.\end{aligned}\quad (5.22)$$

The sound speed vanishes only if  $q, \nu = 1$ , otherwise the pressure gradient acts as a source term in the LHS. Since it is proportional to  $k^2$ , one expects it to have a significant impact to the perturbation growth at subhorizon scales. Indeed, for  $c_K^2$  of order one drastic modifications appear: if the sound speed is negative, the perturbations grow explosively, if it is positive, they will oscillate rather rapidly. Such phenomena are not of course observed in the LSS. Also the ISW effect would be strongly amplified. To alleviate these effects, a decomposition of the unified fluid into two distinct but interacting pieces has been introduced. However, we saw that such decomposition itself does not have any observable consequences, since gravity does not distinguish in what we might label as  $\rho_c$  or  $\rho_X$ . However, by inserting fluctuations in the interaction rate one could perhaps devise a situation where the RHS of Eq.(5.22) would vanish exactly at all times. This would indeed remove the difficulty with structure formation, but requires an ansatz for entropy, that for example in the MPC case amounts to setting

$$\delta T_\nu = -\frac{3\alpha A B a^{3(1+\alpha)}}{[A + B a^{3(1+\alpha)}]^{\frac{1+2\alpha}{1+\alpha}}} H \delta_{0,\nu} \delta_K, \quad (5.23)$$

Only then will entropy perturbations completely silence the effective sound speed in these models, as have been assumed to happen in many studies (see for example Refs.[132, 133, 134, 135] and case I in [1]). As the separation into interacting fluids did not yield an explanation to why this should come about, the question arises why to introduce a decomposition in the first place, since it takes away much of the appeal of the unified approach to the dark cosmology<sup>1</sup>. In the next subsection our approach is instead to study a general cosmic fluid, allowing internal degrees of freedom which could in fact result not only in entropy but also in shear perturbations.

### 5.1.2 Case III: assuming extra dimensions

We noted that there exists another logical possibility, namely to prescribe independent conservation to both  $\rho_X$  and  $\rho_c$ . While the background energy density of the former being still a function of the latter, it might seem like this set-up, where the evolution of these components is interdependent without any explicit physical coupling between them, could be motivated but on astrological grounds, it actually turns out to describe well-defined brane scenarios granted just a certain assumption about their effective four-dimensional gravity. So we begin by assuming that there is just one kind of (cold dark or baryonic) matter,  $\rho_c$ , but due to presence of extra dimensions the effective matter density appearing in the Friedmann equation is  $\rho_K$ .

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<sup>1</sup>In order to satisfy Eq.(5.23) one could introduce yet a third field, which is subdominant at the background but determines the interaction rate.

Now fluctuations in both the physical and the effective matter obey their continuity and Euler equations (4.27) and (4.28). Since we consider a case that the former are driven by the latter, their density perturbations are related adiabatically,

$$\delta_c = \left( \frac{d \log \rho_K}{d \log \rho_c} \right)^{-1} \delta_K = \frac{\delta_K}{1 + w_K} \quad (5.24)$$

Let us proceed in the synchronous gauge, since in such particular choice of gauge the velocity perturbations in matter, and thus also in the effective matter can be set to zero ( $\theta_K$  in any gauge is proportional to  $\theta_c$ ). In this gauge the dark matter continuity equation is also particularly simple,

$$\dot{\delta}_c = -\frac{\dot{h}}{2}, \quad (5.25)$$

and inverting Eq.(5.24) one gets the density perturbation of the effective fluid. Now we should check the consistency with the effective fluid conservation equations. The continuity equation gives

$$\theta_K = -\frac{\dot{h}}{2} - 3H \frac{c_K^2 - w_K}{1 + w_K} \delta_K - \frac{\dot{\delta}_K}{1 + w_K} = 0, \quad (5.26)$$

Since by Eq.(5.24)  $\dot{\delta}_K = 3H(w_K - c_K^2)\delta_K + (1 + w_K)\dot{\delta}_c$ , the RHS vanishes identically. The Euler equation then tells us that there now is anisotropic stress in the effective fluid, and that it is proportional to the density perturbation evaluated in the synchronous gauge:

$$\sigma_K = \frac{c_K^2}{1 + w_K} \delta_K = c_K^2 \delta_c = \frac{\nu B \rho_c^{-q\nu} [(\nu - 1) B \rho_c^{-q\nu} + q\nu - 1]}{[1 + (1 - \nu) B \rho_c^{-q\nu}]^2} \delta_K. \quad (5.27)$$

The last equality holds only for the MPC case, given here as an example. Using gauge transformations (4.9) and (4.11) we find that the Bardeen potentials defined in Eq.(4.17) are now

$$\Phi = \frac{4\pi G a^2}{k^2} \rho'_K(\rho_c) \rho_c \delta_c, \quad \Psi = \frac{4\pi G a^2}{k^2} [\rho'_K(\rho_c) \rho_c - 3\rho''_K(\rho_c)] \delta_c, \quad (5.28)$$

where the linear matter inhomogeneity is given as a solution to the equation

$$\ddot{\delta}_c + 2H\dot{\delta}_c = 4\pi G [\rho'_K(\rho_c) + 3\rho_K \rho''_K(\rho_c)] \delta_c, \quad (5.29)$$

and if the effective matter inhomogeneity is needed, the adiabaticity relation (5.24) is to be consulted. A remarkable property of this simple system is that Eq.(5.29) features only a  $k$ -independent source term: no oscillations or blow-ups occur, although  $\mathcal{S}$  is again zero (still, mainly due to a large ISW at least the MPC parameterization seems to be tightly constrained also in this case [1]). The reason is that now there is effective shear stress (in the next subsection we will consider in detail the effects from such stress).

In Ref.[136] the growth of density perturbations was studied in a universe described by a modified Friedmann equation. Spherical inhomogeneities at subhorizon scales were considered. The authors reasoned that if the hypothetical theory of gravity resulting in

the modified Friedmann equation still respects the Jebsen-Birkhoff law<sup>2</sup>, the evolution of the Schwarzschild-like metric due to the overdensity can be found out by matching that metric with the cosmological one, evolution of which one then gets from the modified Friedmann equation. Since then a dust sphere in a homogeneous universe evolves just like if all matter outside it was removed, one can see what are the conditions for the geodesics of the Schwarzschild-like metric outside the sphere and those of the FRW-like metric inside it to join smoothly. These conditions determine the evolution of the boundary, and since that boundary encompasses a constant mass, of the spherical overdensity  $\delta$ . The result is Eq.(5.29). Then one may proceed more straightforwardly to determine the gravitational potentials. Outside the dust sphere, one has a perturbed FRW metric, and inside an overdense FRW metric with curvature determined by Eq.(5.29). Matching again the two metrics gives now the gravitational potentials, and they come out as in Eq.(5.28).

Thereby we have seen that the condition for legitimacy of our approach to treat the corrections as driven by standard matter is at subhorizon scales equivalent with the validity of the Jebsen-Birkhoff theorem. On the other hand, an approach where the superhorizon perturbations in non-standard cosmologies are assumed to be governed by the background equations [138] also results in the same evolution equation (5.29) for the inhomogeneities we found. These results can be understood as follows. In our treatment  $\mathcal{S}_K$  is vanishing, and in the absence of entropy there are no new effective degrees of freedom. Thus the background determines the perturbation evolution uniquely (leading to the large-scale solution of Ref.[138]) as well as the two pieces of information, mass and spherical symmetry of a source, are sufficient to determine the metric outside the source uniquely (leading to the small-scale solution of Ref.[136]).

In the particular but interesting case of DGP gravity the Jebsen-Birkhoff theorem is known to be not respected. Many efforts have been undertaken to determine the growth of perturbations in that model. We regard Koyamas contribution in Ref.[139] an important recent development in that field, since he, in addition to showing that previous attempts had been unsatisfactory since the perturbation equations that had been derived violated the Bianchi identities (in our terms, Eq.(5.11) did not hold), constructed covariant effective field equations that would yield the desired modified expansion law, and could be used to compute also the perturbation equations. These effective field equations read

$$G_{\mu\nu} = -\frac{1}{4}\tilde{T}_{\mu\alpha}\tilde{T}_\nu^\alpha + \frac{1}{12}\tilde{T}_\alpha^\alpha + \frac{1}{24}\left[3\tilde{T}_{\alpha\beta}\tilde{T}^{\alpha\beta} - (\tilde{T}_\alpha^\alpha)^2\right]g_{\mu\nu} - E_{\mu\nu}, \quad (5.30)$$

where  $\tilde{T}_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu}$ , and  $E_{\mu\nu}$  is a projection of the 5-dimensional Weyl tensor. Since one knows about  $E_{\mu\nu}$  only that it is vanishing at the background order and that it is traceless, the perturbation equations cannot yet be uniquely determined. In the case that  $E_{\mu\nu} = 0$ , our Eq.(5.29) follows again. This makes sense: if the tensor vanishing at background does not affect perturbations, there is no entropy in the effective fluid. The problem that  $E_{\mu\nu}$  has to be guessed is, in our approach which is applicable to any form of a modified Friedmann equation, translated into the problem of not knowing the right ansatz for entropy. This simple and physically intuitive approach allows to make contact with other interesting

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<sup>2</sup>What has been known as the Birkhoff theorem of general relativity was first probed by the Norwegian J. T. Jebsen in 1921. For more historical details, see Ref.[137]. The theorem states that for any particle outside a spherically symmetric mass, the metric observed by that test particle is equivalent to that of a point source of the same mass located at the center of the sphere.

considerations, and as demonstrated here in the DGP case, it might prove to be a useful starting point to study even such models where some of the underlying assumptions would have to be relaxed.

## 5.2 Fluid dark energy

In its simplest descriptions the dark energy component is described fully by an EoS. When considering perturbations, a more detailed description of a cosmological fluid is necessary. The energy momentum tensor can then be written as in Eq.(4.3), where  $\Pi_{\mu\nu}$  can include only spatial inhomogeneity. We define perfect fluid by the condition  $\Pi_{\mu\nu} = 0$ . If in addition the fluid is adiabatic,  $p = p(\rho)$ , the evolution of its perturbations is described by the adiabatic speed of sound  $c_a$ . This is in turn fully determined by the EoS  $w$ ,

$$c_a^2 \equiv \frac{\dot{p}}{\dot{\rho}} = w - \frac{\dot{w}}{3H(1+w)}. \quad (5.31)$$

For an adiabatic fluid,  $\delta p = c_a^2 \delta \rho$ .

In the general case, there may be more degrees of freedom and the pressure  $p$  might not be a unique function of the energy density  $\rho$ . An extensively studied example is quintessence (recall section 3.2.1). For such scalar fields the variables  $w$  and  $c_s^2$  depend on two degrees of freedom: the field and its derivative, or equivalently, the kinetic and the potential energy of the field. Then the dark energy (entropic) sound speed is defined as the ratio of pressure and density perturbations in the frame comoving with the dark energy fluid,

$$c_s^2 \equiv \frac{\delta p}{\delta \rho|_{de}}. \quad (5.32)$$

In the adiabatic case,  $c_s^2 = c_a^2$ , which holds in any frame, but in general the ratio  $\delta p / \delta \rho$  is gauge dependent. Hence, in the case of entropic fluid such as scalar fields, one needs both its equation of state and its sound speed as defined in Eq. (5.32) to describe dark energy.

However, in order to have an even more general set of parameters to fully describe a dark energy fluid and its perturbations, besides  $w$  and  $c_s$ , one should also consider the possibility of anisotropic stress. This is important because it enters directly into the Newtonian metric, as opposed to  $w$  and  $c_s$  which only contribute through the causal motion of matter [140]. Taking this generalization into account, in the synchronous gauge Eqs.(4.15,4.16), the evolution equations for the dark energy density perturbation and velocity potential, Eqs.(4.27) and (4.28) can be written as

$$\dot{\delta} = -(1+w) \left\{ [k^2 + 9H^2(c_s^2 - c_a^2)] \frac{\theta}{k^2} + \frac{\dot{h}}{2} \right\} - 3H(c_s^2 - w)\delta, \quad (5.33)$$

$$\dot{\theta} = -H(1 - 3c_s^2)\theta + \frac{c_s^2 k^2}{1+w}\delta - k^2\sigma, \quad (5.34)$$

where  $h$  is the trace of the synchronous metric perturbation. Here  $\sigma$  is the anisotropic stress of dark energy, related to notation of Eq.(4.3) by  $(\rho + p)\sigma \equiv -(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij})\Sigma^{ij}$ . Basically,

while  $w$  and  $c_s^2$  determine respectively the background and perturbative pressure of the fluid that is rotationally invariant,  $\sigma$  quantifies how much the pressure of the fluid varies with direction.

Generally such a property implies shear viscosity in the fluid, and thus its effect is to damp perturbations. A covariant form for the viscosity generated in the fluid flow is [48]

$$\Sigma_{\mu\nu} = \varsigma \left( u_{\mu;\alpha} h_{\nu}^{\alpha} + u_{\nu;\alpha} h_{\mu}^{\alpha} - \frac{2}{3} u^{\alpha}_{;\alpha} h_{\mu\nu} \right) + \zeta u^{\alpha}_{;\alpha} h_{\mu\nu}. \quad (5.35)$$

Now the conservation equations  $T^{\mu\nu}_{;\mu} = 0$  reduce to the Navier-Stokes equations in the non-relativistic limit. Here  $\varsigma$  is the shear viscosity coefficient, and  $\zeta$  represents bulk viscosity. Here we set the latter to zero since we want  $\Sigma_{ij}$  to be traceless and to vanish in the background, but  $\zeta \neq 0$  would violate both of these requirements. In cosmology we have  $u_{\mu} = a(1, -v_{,i})$  in the synchronous gauge, and the velocity potential  $\theta$  is the divergence of the fluid velocity  $v$ . One can then check that the components of Eq.(5.35) vanish except in the off-diagonal of the perturbed spatial metric, and that the non-vanishing components are given by (see appendix B.1)

$$\sigma = \frac{8\varsigma}{9} \frac{ak}{\rho + p} \left( \theta + \frac{1}{2} \dot{h} + 3\dot{\eta} \right), \quad (5.36)$$

where the two last terms in the RHS is the metric shear constructed from the synchronous gauge potentials, Eq.(4.16). We mean that  $-k^2\chi/a = -h/2 - 3\eta$ . From the coordinate transformation properties of  $T_{\mu\nu}$  it follows that  $\sigma$  must be gauge-invariant, and indeed the linear combination in the LHS is frame-independent.

However, the anisotropic stress is not necessarily given directly as in Eq.(5.36). For neutrinos this term instead acts as a source for the anisotropic stress, which is also coupled to higher multipoles in the Boltzmann hierarchy. Thus the evolution of the stress must, at least in principle, be solved from a complicated system of evolving multipoles. The approach we will use in this thesis to specify the shear viscosity of the fluid is more in line with the neutrino stress than Eq.(5.36). Following Hu [140], we describe the evolution of the anisotropic stress with the equation

$$\dot{\sigma} + 3H \frac{c_a^2}{w} \sigma = \frac{8}{3} \frac{c_{vis}^2}{1+w} \left( \theta + \frac{\dot{h}}{2} + 3\dot{\eta} \right). \quad (5.37)$$

Then the shear stress is not determined algebraically from fluctuations in the fluid as was the case in Eq. (5.36), but instead it must be solved from a differential equation.

This phenomenological set-up is motivated as follows [140]. One can guess that the anisotropic stress is sourced by shear in the velocity and in the metric fluctuations. Again one must take into account the gauge-invariance  $\sigma$  in constructing a source term in the differential equation. As mentioned, an appropriate linear combination appears in LHS of Eq.(5.36). Up to the viscosity parameter  $c_{vis}^2$ , this determines the right hand side of Eq. (5.37). In the left hand side there appears also a drag term accounting for dissipative effects. We have adopted a natural choice for the dissipation time-scale,  $\tau_{\sigma}^{-1} = 3H$ . One may then check that Eq. (5.37) with  $w = c_{vis}^2 = 1/3$  reduces to the evolution equation for the massless neutrino quadrupole in the truncation scheme where the higher multipoles are neglected [120] (this applies also to photons when one ignores their polarization and



coupling to baryons). In what follows, we will study the consequences of Eq.(5.37) for fluids with negative EoS. For  $w < -1$ , one should consider negative values of  $c_{vis}^2$ , as was suggested in Ref. [141]. So the parameter  $c_{vis}^2/(1+w)$  should remain positive. We will return to this below.

Note that the parameterization of Eqs.(5.33), (5.34) and (5.37) describes cosmological fluids in a very general way. The system reduces to cold dark matter equations when  $(w, c_s^2, c_{vis}^2)$  is  $(0, 0, 0)$  and relativistic matter corresponds to  $(1/3, 1/3, 1/3)$ . A scalar field with a canonical kinetic term is given by  $(w(a), 1, 0)$ , where  $-1 < w(a) < 1$ . With an arbitrary kinetic term one can construct k-essence models (see Table 3.1) characterized by unrestricted equation of states and speeds of sound,  $(w(a), c_s^2(a), 0)$ , but vanishing shear. On the other hand, one should keep in mind that the parameterization cannot be completely exhaustive. It does not cover, for example, a cosmological fluid with anisotropic stress determined by Eq. (5.36) when  $\varsigma \neq 0$ . We might address the viability of this approximation elsewhere, but restrict here to the parameterization given in Eq.(5.37).

We will investigate the effect of dark energy perturbations on the CMB anisotropies and on the matter power spectrum with the simplest assumption that all of the three parameters  $w$ ,  $c_s^2$  and  $c_{vis}^2$  are constant<sup>3</sup>. The CMB and large scale structure in constant  $w$  dark energy has been analyzed in Ref. [143] but without including the anisotropic stress, effects of which we therefore emphasize here. More detailed and extensive treatment can be found in our paper [5].

### 5.2.1 Models with $-1 < w < 0$

We will first consider the case that  $w$  is negative but larger than  $-1$ . The metric perturbation  $h$  is now a source in Eq. (5.33), which tends to draw dark energy into overdensities of cold dark matter. However, for large scales the source due to velocity perturbations is proportional to  $-(1+w)(c_s^2 - w)\theta/k^2$ , and this term can dominate the metric source term and drive  $\delta$  to smaller values. In fact  $\delta$  drops below zero when evaluated in the synchronous gauge<sup>4</sup>. This happens especially for large sound speeds, since then both the friction term in Eq. (5.34) and the source term in Eq. (5.33) are larger (see FIG. 5.1). Thus  $\delta$  gets smaller when dark energy begins to dominate. The ISW effect is enhanced when one increases the sound speed squared. The effect of the anisotropic stress is also to wash out overdensities. This is because the metric part of the source term in Eq. (5.37) turns out negative, and it dominates over the velocity term. Thus  $\sigma$  is driven to negative values, and in Eq. (5.34) it will act to increase the growth of  $\theta$ . This is similar to free-streaming of neutrinos, although for them the effect is relevant at smaller scales. Since now  $c_a^2 < 0$ , the source term  $\sim H^2\theta/k^2$  in Eq. (5.33) inhibits structure growth at large scales. Therefore, as the dark energy becomes dominant, the overall density structure is smaller when  $c_{vis}^2$  is larger, and the ISW effect is amplified.

<sup>3</sup>For the energy content of the universe we use  $\Omega_b = 0.044$ ,  $\Omega_c = 0.236$ ,  $\Omega_{de} = 0.72$  and  $h_0 = 0.68$ . In all the numerical calculations we assume a scale invariant initial power spectra with adiabatic initial conditions. Optical depth to last scattering is set to zero. Although we do not search for the best-fit models here, we include the WMAP data [66] and the SDSS data [71] in to the figures. The calculations are performed with a modified version of the CAMB code [142].

<sup>4</sup>More accurately, we evaluate the transfer functions of the perturbations. A negative value for the transfer function indicates that a perturbation variable acquires the opposite sign to the initial value of  $\mathcal{R}$ .

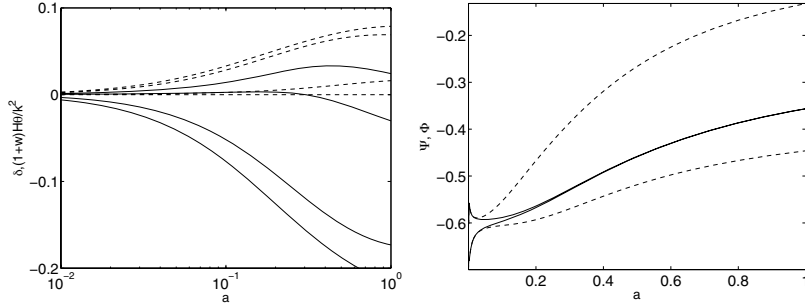


Figure 5.1: In synchronous gauge (left panel): Late evolution of the dark energy density perturbation and velocity potential for  $k = 1.3 \cdot 10^{-4} \text{ Mpc}^{-1}$  when  $w = -0.8$ . Solid lines from top to bottom correspond to  $\delta$ , and dashed lines from bottom to top correspond to  $(1+w)H\theta/k^2$  when  $(c_s^2, c_{vis}^2) = (0,0), (0.6,0), (0,0.6), (0.6,0.6)$ . The effect of  $c_{vis}^2$  is to damp density perturbations, which in the synchronous gauge is seen as a consequence of enhancing the velocity perturbations.

In the Newtonian gauge (right panel): Late evolution of the gravitational potentials at large scales ( $k = 1.3 \cdot 10^{-4} \text{ Mpc}^{-1}$ ) when  $w = -0.8$  and  $c_s^2 = 0$ . Solid lines are for the case of perfect dark energy and dashed for the imperfect case with  $c_{vis}^2 = 1.0$ . The upper lines are  $\Phi$ , the lower lines are  $\Psi$ .

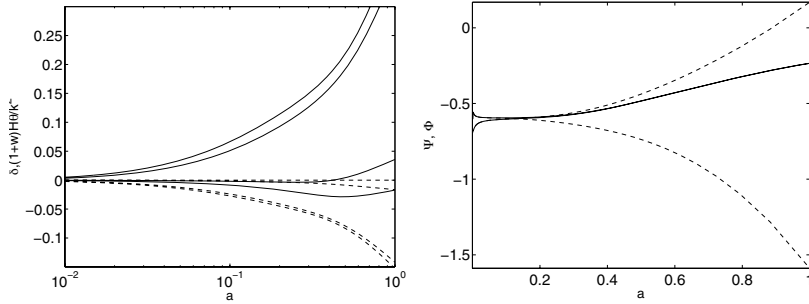


Figure 5.2: In the synchronous gauge (left panel): Late evolution of the dark energy density perturbation and the velocity potential for  $k = 1.3 \cdot 10^{-4} \text{ Mpc}^{-1}$  when  $w = -1.2$ . Solid lines from bottom to top correspond to  $\delta$ , the dashed lines from top to bottom correspond to  $(1+w)H\theta/k^2$  when  $(c_s^2, c_{vis}^2) = (0,0), (0.6,0), (0,-0.6), (0.6,-0.6)$ . The effect of  $c_{vis}^2$  is to increase clustering, which in the synchronous gauge is seen as a consequence of enhancing the velocity perturbations.

In the Newtonian gauge (right panel): Late evolution of the gravitational potentials at large scales ( $k = 1.3 \cdot 10^{-4} \text{ Mpc}^{-1}$ ) when  $w = -1.2$  and  $c_s^2 = 0$ . Solid lines are for the case of perfect dark energy and, dashed for the imperfect case with  $c_{vis}^2 = -1.0$ . The upper lines are  $\Psi$ , the lower lines are  $\Phi$ .

It is illuminating to describe the same thing also in the Newtonian gauge, Eq.(4.17). We remind that the ISW effect stems from the time variation of the metric fluctuations,

$$C_\ell^{ISW} \propto \int \frac{dk}{k} \left[ \int_0^{\tau_{LSS}} d\tau (\dot{\Phi} + \dot{\Psi}) j_\ell(k\tau) \right]^2, \quad (5.38)$$

where  $\tau_{LSS}$  is the conformal distance to the last scattering surface and  $j_\ell$  the  $\ell$ 'th spherical Bessel function. The ISW effect occurs because photons can gain energy as they travel through time-varying gravitational wells. These wells are in turn caused by matter, since

$$-k^2\Psi = 4\pi G a^2 \rho \left[ \delta + 3 \frac{H}{k^2} (1+w)\theta \right]_{|T}. \quad (5.39)$$

We have indicated with the subscript  $|_T$  that in the left hand side variables refer to all matter present, and not just dark energy. Note also that the term in square brackets is gauge-invariant. Thus, evaluated in any frame, it equals  $\delta_v$ , the overdensity of energy seen in the comoving frame (recall the Poisson Eq.(5.21)). During matter domination,  $\delta_c$  grows in such a way that the gravitational potentials stay constant. It is then clear that as dark energy begins to take over, the gravitational potential  $|\Psi|$  begins to decay. Contrary to expectations from FIG. 5.1, this decay is not more efficient at large scales when there is shear, as shown in FIG. 5.1. This is because the dark energy shear influences gravitational wells in such a way that the growth of matter perturbations does not slow down as much as in a perfect universe.

However, there is an important twist to the story. This is seen in the FIG. 5.1, where the evolution of the potentials  $\Phi$  and  $\Psi$  is plotted at very large scales. At an early time the potentials are unequal because of the free streaming of radiation. However, our attention is now on the late evolution of the potentials. Due to dark energy, the potentials can redepert from each other at smaller redshifts. This can happen only when  $c_{vis}^2 \neq 0$ , since

$$\Phi = \Psi - \frac{3a^2}{2M^2} (1+w) \rho \sigma_{|T}, \quad (5.40)$$

i.e. shear is the difference between the depth of matter-induced gravity well and the amount of spatial curvature. Since  $\sigma$  is gauge-invariant, and we found that it becomes negative for dark energy, we can see that shear perturbation drives  $|\Phi|$  to vanish more efficiently. Thereby we find that the effect of shear on Eq.(5.39) only partly compensates for the effect on  $\Phi$  from Eq.(5.40), and thus the overall ISW from Eq.(5.38) will be amplified when dark energy perturbations tend to smooth as in FIG. 5.1.

In FIG. 5.3 we show the large angular scales of the CMB spectrum when  $w = -0.8$  and the two other parameters are varied. The left panel depicts the case where the sound speed of dark energy vanishes. Then the pressure perturbation vanishes and the clustering of dark energy is inhibited only by the free-streaming effect of shear viscosity. Therefore the large scale power of the CMB is increased due to the ISW effect by increasing  $c_{vis}^2$ . In the panel panel  $c_s^2 = 1$ . Then dark energy is almost smooth (except at the largest scales) even without anisotropic stress, and thus we see a smaller effect when  $c_{vis}^2$  is increased. When  $c_{vis}^2 < 0$  the metric and the fluid sources drive the perturbations in the same direction, resulting in explosive growth. Since this would spoil the evolution except when  $c_{vis}^2$  is tuned to infinitesimal negative values, we will not consider such a case here.

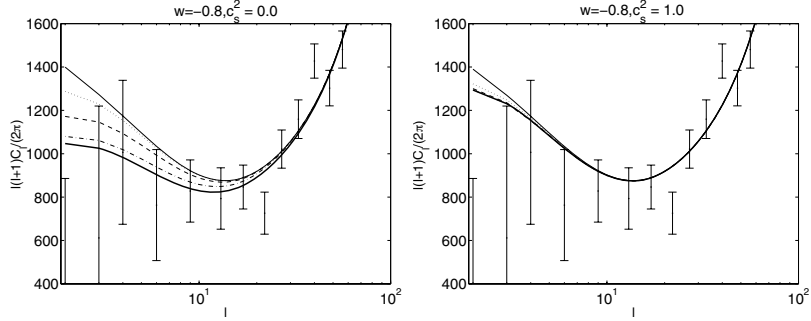


Figure 5.3: The CMB anisotropies for  $w = -0.8$ . In the left panel  $c_s^2 = 0$  and in the right panel  $c_s^2 = 1.0$ . The ISW contribution increases with the parameter  $c_{vis}^2$ : thick lines are for  $c_{vis}^2 = 0$ , dash-dotted for  $c_{vis}^2 = 0.001$ , dashed for  $c_{vis}^2 = 0.01$ , dotted for  $c_{vis}^2 = 0.1$  and the solid lines for  $c_{vis}^2 = 1.0$ . There is a small effect when vanishing of dark energy sound speed allows it to cluster. Otherwise the effect is negligible.

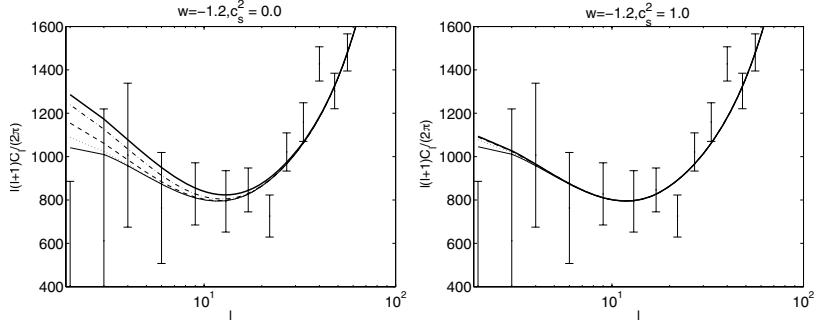


Figure 5.4: The CMB anisotropies for  $w = -1.2$ . In the left panel  $c_s^2 = 0$  and in the right panel  $c_s^2 = 1.0$ . The ISW contribution decreases with the parameter  $c_{vis}^2$ : thick lines are for  $c_{vis}^2 = 0$ , dash-dotted lines for  $c_{vis}^2 = 0.001$ , dashed lines for  $c_{vis}^2 = 0.01$ , dotted lines for  $c_{vis}^2 = 0.1$  and the solid lines for  $c_{vis}^2 = 1.0$ . Again one can see a small effect when dark energy clusters somewhat. If it has sound speed equal to one, the clustering is inhibited and the effect from shear is negligible. The shear now reduces instead of increasing the late ISW effect contribution.

### 5.2.2 Phantom models with $w < -1$

When the dark energy EoS is less than  $-1$ , the effect of both the sound speed and of the viscosity parameter are the opposite to the previous case. Now the source term in Eq. (5.33) has its sign reversed, and because of that dark energy falls out from the overdensities. Similarly, the velocity potential acts now as a source for the overdensities. Therefore increasing the sound speed will drive dark energy to cluster more efficiently. Now the effect of  $c_{vis}^2 > 0$  is with the same sign of those of the metric sources, and therefore we must consider negative values for this parameter. Then, if we increase the parameter  $c_{vis}^2/(1+w)$ , the dark energy perturbations are growing more efficiently, as shown in FIG. 5.2. This is because  $\sigma$  is negative, just like in the previous case, and again tends to enhance the velocity potential. The crucial difference in the perturbation evolution for imperfect dark energy here as compared to the imperfect  $w > -1$  case is that more shear in the perturbations will result in more clumpy structure in the density of phantom dark energy.

One can again consider the ISW in terms of the Newtonian gauge potentials, Eq.(5.38). The effect is not directly seen from the behaviour of  $\delta$  in FIG. 5.2, partly because of the different gauge and partly because the anisotropic stress induces a compensation on the other gravitational potential and thereby influences also the matter perturbation. This is shown in FIG. 5.2. The anticipated simple result (that the decay of the gravitational potentials is reduced since  $\delta$  is enhanced when there is more shear) again holds for the sum of the gravitational potentials, but considering  $\Psi$  or  $\Phi$  separately reveals the intricacy of the fluctuation dynamics due to anisotropic stress. Again the evolution of  $\Psi$  implies, through Eq.(5.39), that the influence of  $\sigma$  to dark matter is the opposite from dark energy, but on the other hand, evolution of the spatial curvature  $\Phi$  implies that the sum  $\Phi + \Psi$  behaves according to the dominating component. Now the gravitational well  $\Psi$  grows deeper, because the contribution from shear in the phantom fluid in Eq.(5.40) comes with a minus sign.

In FIG. 5.4 we have plotted the large angular scales of the CMB spectrum when  $w = -1.2$  and the two other parameters are varied. The left panel depicts the case that the sound speed of dark energy vanishes. Then the ISW effect without anisotropic stress is large since dark energy perturbations are nearly washed out. Consequently, the large scale power of CMB is decreased as  $|c_{vis}^2|$  is increased, since the "anti-viscosity" will then amplify perturbations. In the right panel  $c_s^2 = 1$ . There the effect of  $c_s^2$  already dominates, and we see a smaller difference when  $c_{vis}^2/(1+w)$  is increased.

### 5.2.3 Models with $c_s^2 < 0$

For perfect dark energy models without shear, the case  $c_s^2 < 0$  leads to explosive growth of perturbations. This is analogous to the behaviour of a simple wave, which has a solution  $\sim e^{-i(k/c_s)t + i\vec{k}\cdot\vec{x}}$ , diverging when the sound velocity is imaginary. It is not clear, however, how useful the analogy to the sound speed of a simple plane wave is to the interpretation of the variable defined by Eq. (5.32). For instance, in the modified gravity context [1] this formal definition does not describe propagation of waves in any physical matter. A priori one should not discard the possibility  $c_s^2 < 0$  without careful deliberation (related issues will be discussed in Section 6.1). In fact, given a fluid with negative EoS, one would expect, from Eq. (5.31), also a negative sound speed squared. To get rid of this feature, extra degrees of freedom must be assumed to exist in such a way that the variable defined

$w$	$c_s^2$	$c_{vis}^2 < 0$	$c_{vis}^2 = 0$	$c_{vis}^2 > 0$
$> -1$	$> 0$	diverges	canonical scalar field	$\searrow$ (FIG. 5.3)
	$< 0$	diverges	diverges	$\nearrow \dagger[5]$
$< -1$	$> 0$	$\nearrow$ (FIG. 5.4)	phantom scalar field	diverges
	$< 0$	$\searrow \dagger [5]$	diverges	diverges

Table 5.1: Summary of different parameter regions for dark energy fluids. We have indicated with  $\nearrow$  the cases where superhorizon perturbations are increased as  $|c_{vis}^2|$  is increased, and with  $\searrow$  the cases where superhorizon perturbations in dark energy are smoothed as  $c_{vis}^2$  is increased. We indicate by  $\dagger$  that the shear perturbation influences significantly also the small scale perturbations.

by Eq. (5.32) turns out positive.

When the generation of shear in the fluid is taken into account, the perturbation growth for  $c_s^2 < 0$  can be stabilized. This is because shear is sourced by the perturbations, and in turn the shear will inhibit clustering. Here it is possible to choose the parameters in such a way that the dark energy perturbation grows steadily at late times. So the ISW effect comes with the opposite sign from the Sachs-Wolfe effect, which leaves its imprint in the CMB earlier. These effects cancel each other and thus the large scale power in the CMB spectrum is reduced, in accordance with the measured low quadrupole.

#### 5.2.4 A summary

We summarize the features of dark energy perturbations in different parameter regions in Table 5.1. A half of the parameter space is excluded because divergent behaviour occurs, and much of the remaining parameter space is degenerate. Even when restricting to the simplest case where the parameters are kept constant, it seems clear that present observational data allows a large variety of interesting models with non-vanishing shear,  $c_{vis}^2 \neq 0$ . In some parameter regions of Table 5.1 new features appear at observable scales.

The dark energy shear does not leave a discernible imprint to the matter power spectrum of cold dark matter and baryons. The effect occurs only at scales much larger than what current observations are able to probe. In the spectrum of the total density perturbation one would see more pronounced features at scales more tantalizingly near the current limits of observations. However, there is no way to directly measure the dark energy density perturbation. In FIG. 5.1 and FIG. 5.2 it was seen that the shear changes the Newtonian gravitational potentials significantly. Thus one might hope to find a way to study whether effects from an anisotropic stress could be measured by using for example the cross correlation of the ISW signal and the large scale structure observations or gravitational lensing experiments. However, one should keep in mind that we have considered perturbations at vast scales. We have found that fluctuations in an imperfect as well as in a perfect fluid with a constant  $w < -1/3$  are confined to superhorizon scales. This is except for special occasions, in particular when the parameter  $c_s^2$  is negative or when the perturbations behave pathologically due to wrong sign of the viscous parameter. For the largest scales, the viscous parameter determines the evolution of the perturbations. One can note that the variation of the sound speed  $c_s^2$  has much less effect on the evolution

of perturbations in the limit  $k \rightarrow 0$ . However, the parameter  $c_s^2$  sets the scale at which the fluctuations in dark energy become negligible. For smaller  $c_s^2$ , there are fluctuations at smaller wavelengths. Therefore the shear would be best seen when the  $c_s^2$  is nearly zero or even negative.

To conclude, the main impact of dark energy anisotropic stress on observations seems to be the modification of the CMB at very large scales, from which it would be very difficult to unambiguously detect due to cosmic variance. However, even then  $c_{vis}^2$  can bias measurements of other parameters, like  $w$  or  $c_s^2$ . Also, as we have demonstrated in Ref.[5], dark energy shear can have detectable effects when one considers the perhaps better physically motivated situation where the parameters  $w$ ,  $c_s^2$  and  $c_{vis}^2$  are allowed to evolve in time.

# Chapter 6

## Models

In this section we will discuss some specific models of dark energy, and their predictions to some specific observations. Firstly we will look at the possibility of a scalar field coupling to the Gauss-Bonnet invariant. Implications of such coupling to the stability and the evolution of small scale perturbations will be our main concern. Secondly, we will mention the possibility of probing dark energy models with the ISW-LSS correlation. A quintessence model with isocurvature initial conditions will then be considered in a bit more detail. Lastly, perturbation evolution with modified matter sources in alternative theories of gravity featuring second order field equations will be discussed.

### 6.1 Gauss-Bonnet gravity: Stability of perturbations

If one wants to consider a unique gravitational action generating second order field equations, the Gauss-Bonnet (GB) invariant should be there by default. The GB term has a desirable feature which is called quasi-linearity: the highest derivatives of the metric appear in the field equations only linearly, so as to make the theory ghost free (at least in constant curvature backgrounds, like Minkowski). Interestingly, the GB term is the unique combination of the curvature squared terms with this property. This quadratic invariant constructed from the metric as

$$R_{GB}^2 \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2. \quad (6.1)$$

This particular combination appears frequently in attempts at quantum gravity, especially in stringy set-ups. In fact, all versions of string theory in 10 dimensions (except Type II) include this term as the leading order  $\alpha'$  correction [144, 145]. In four dimensions the Gauss-Bonnet term is a total divergence, and thus its appearance alone in the action can be neglected for all practical purposes. However, in several occasions its presence may lead to contributions to the field equations in four dimensions. Specifically, the Gauss-Bonnet term could be non-minimally coupled to a scalar field. Then a partial integration of the action gives the vanishing boundary term but in addition, because of the field dependence of the coupling, a non-vanishing term involving the integral of  $R_{GB}^2$  which is of the first order by construction. Explicitly, one then finds

$$T_{\mu\nu}^{GB} = 8 \left[ -f_{;\alpha\beta} R_{(\mu}^{\alpha\beta}{}_{\nu)} - \square f R_{\mu\nu} + 2f_{;\alpha(\mu} R_{\nu)}^{\alpha} - \frac{1}{2} f_{;\mu\nu} R \right] - 4 \left[ 2f_{;\alpha\beta} R^{\alpha\beta} - \square f R \right] g_{\mu\nu}, \quad (6.2)$$



where  $f(\phi)$  is the coupling term.

Thus the action for the system to consider is

$$S = \int d^4x \sqrt{-g} \left[ \frac{M^2 R}{2} - \frac{\gamma}{2} (\nabla \phi)^2 - V(\phi) - f(\phi) R_{GB}^2 + L_m \right],$$

where  $\gamma$  is a constant and  $f(\phi)$  is a coupling<sup>1</sup>. It is natural to include the coupling  $f(\phi)$  in the action. It appears, for example, in the one-loop corrected string effective action [146, 147] when going from the string (Jordan) frame to the Einstein frame. In FRW background the action (2.16) yields the Friedmann equation

$$3M^2 H^2 = \frac{\gamma}{2} \dot{\phi}^2 + a^2 V(\phi) + \rho_m + 24a^{-2} H^3 f'(\phi) \dot{\phi}, \quad (6.3)$$

and the Klein-Gordon equation

$$\gamma(\ddot{\phi} + 2H\dot{\phi}) + a^2 V'(\phi) + 24a^{-2} f'(\phi) H^2 \dot{H} = 0, \quad (6.4)$$

In our numerical examples, we adopt an exponential form for the potential and the coupling,

$$V(\phi) = V_0 e^{-\lambda \phi / (\sqrt{2}M)}, \quad f = f_0 e^{\alpha \phi / (\sqrt{2}M)}. \quad (6.5)$$

The nonperturbative effects from gaugino condensation or instantons can result in an exponential potential. There is also phenomenological motivation for this, since, besides its being a simple choice, it has been recently shown that even in the presence of the Gauss-Bonnet coupling, a canonic scalar field can exhibit scaling behaviour only if its potential is exponential[148]. An exponential field-dependence for coupling can be a good approximation in supergravity actions (for massless dilaton one in fact has  $f(\phi) = \sum C_n e^{(n-1)\phi}$  [149]). In addition,  $f$  indeed should grow fast, if we want the Gauss-Bonnet term to have a significant impact to the late cosmology, while it is negligible in the past. Finally, these exponential forms allow to consider natural scales for all the parameters appearing in the model: if  $\phi \sim 400/\alpha$  at the present, we have the dimensionless constant  $f_0$  of order one, and simultaneously the potential scale  $V_0$  (with dimensions of energy density) of the order  $V_0 \sim M^4$  (however, the mass of the field, formally defined as  $m_\phi^2 = V''(\phi)$ , will turn out to be very small, as usual in quintessence models). Also the exponents  $\alpha$  and  $\lambda$  are of order one (and both positive). We presented details of possible cosmologies with the parameterization (6.5) in Refs.[7, 150]. There we showed that such models can produce plausible expansion history, with a possible transient phantom era. Constraints from cosmological observations were computed (a summary with the baryon oscillation data added is in Fig.(6.1)), and implications for the perturbation theory were brought up in this context.

Here we will review some interesting aspects of the latter. It is possible to derive an evolution equation similar to Eq.(5.22) for matter perturbations in Gauss-Bonnet gravity, though it is useful only at small scales,  $aH \ll k$  (otherwise being much too complicated). One arrives at

$$\ddot{\delta} + H\dot{\delta} = 4\pi G_* a^2 \rho \delta. \quad (6.6)$$

---

<sup>1</sup>In string theory,  $f(\phi) = \sigma - \hat{\delta}\xi(\phi)$ : the coupling  $\sigma$  may be related to string coupling  $g_s$  via  $\sigma \sim 1/g_s^2$ , and the numerical coefficient  $\hat{\delta}$  typically depends on the massless spectrum of every particular model [15].

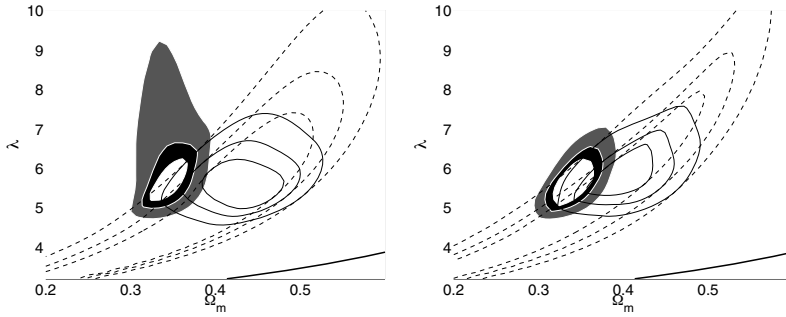


Figure 6.1: The 68, 90 and 99 percent confidence limits for the model in the  $\Omega_m - \lambda$  plane. The left panel is with Gold data and  $\alpha$  is marginalized over in the range  $1.5\lambda < \alpha < 10\lambda$ , the right panel is with SNLS data with optimal  $\alpha$ . Dashed lines are constraints from the SNeIa data, solid lines from the combined SNeIa and CMB shift parameter data, and the solid contours include in addition the baryon oscillation scale. Below the line the scaling solution is unstable regardless of  $\alpha$  and tuning of initial conditions is necessary: elsewhere initial conditions do not matter (and acceleration occurs whenever  $\alpha > \lambda$ ). If the scalar field is tracking already in the nucleosynthesis epoch, there is a tension with the amount of early quintessence and the nucleosynthesis limit. Conservatively, this limit translates to  $\lambda > 6.3$  [29]. With the SNeIa and CMB shift parameter data, the fit is slightly better than in the  $\Lambda$ CDM, even when taking into account that the model has a couple of extra parameters when compared with  $\Lambda$ CDM. When baryon oscillation constraint is included, the fit is worse. However, this constraint is derived assuming the  $\Lambda$ CDM model, and presently it is unclear whether one can apply it as such in alternative cosmologies such as the one considered here.

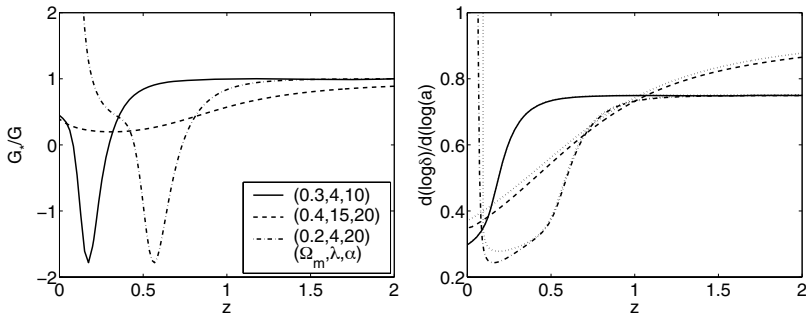


Figure 6.2: Evolution of effective gravitational constant  $G_*$  (left panel) and of the dimensionless growth rate  $d \log \delta_m / (d \log a)$  (right panel) as functions of redshift. We see that divergence is possible for some parameter combinations. The thick lines in the right panel are numerical solutions of the full linearized equations, and the dotted lines are solutions to the approximative equation (6.6). The agreement is excellent in most of the parameter space, though in the two extreme cases depicted here (dashed line corresponding to very slow transition, the dash-dotted line corresponding to instability) deviation is visible.

The effective gravitational constant  $G_*$  felt by the matter inhomogeneities depends on the parameters of the model. Explicitly, it is given by

$$\frac{G_*}{G} = 4 \frac{-x^4 + \mu^2(1 + \epsilon)^2 + x^2[2(1 + \epsilon)(\mu - 1) + y]}{x^2[4 + \mu(5\mu - 8)] - \mu^2[6(1 + \epsilon)(\mu - 1) + y]}, \quad (6.7)$$

where we have defined the dimensionless variables  $x = \dot{\phi}/(\sqrt{2}MH)$ ,  $\mu = 8a^2\dot{\phi}Hf'(\phi)/(M^2H^2)$ ,  $y = V(\phi)/(M^2a^2H^2)$  and the slow roll parameter  $\epsilon = \dot{H}/H^2$ . We can observe several interesting things by staring at Eq.(6.6).

Firstly, the shape of the matter power spectrum is the same as for  $\Lambda$ CDM cosmology. The superhorizon scales, where our approximation breaks down and the density perturbation becomes gauge dependent, are not efficiently probed by the present large scale structure surveys. As we have shown that at subhorizon scales the growth rate of structure is the same at all scales, which also the case when the acceleration is driven by vacuum energy, we cannot distinguish between these two very different scenarios by comparing the shape of matter power spectrum (assuming of course that the primordial spectrum is the same in both of the cases). Therefore the primary constraints arising from the matter power spectrum could be deduced from the overall normalization. Note that these conclusions are model independent, the only assumption being that the scalar field is not very massive. We did not make specific assumptions on the coupling or the potential or make the approximation that  $\Omega_m$  dominates in order to deduce these results. The constraints from large scale structure are still of course relevant, since it is possible that for some models the (scale-independent) growth rate is considerably modified and the normalization of the power spectrum can be used to exclude such cases. We plot the evolution of the dimensionless growth rate  $(d \log \delta_m)/(d \log a)$  and of  $G_*$  in our example model in FIG.(6.2).

Secondly, the stability of the model can be read off from the expression (6.7). It can in principle diverge. Indeed, we have found that in the example model Eq.(6.5),  $G_*$  typically diverges in the future, and for low matter densities this can happen even before  $a = 1$ . It is unclear what happens at such point, since the linear approximation certainly breaks down near the (what would be) singularity. Perturbations will at least for a while grow explosively, but possibly this will not lead a finite time big rip<sup>2</sup>. It is tempting to believe that since the linear matter perturbations grow so sizable that the FRW description breaks down, some kind of matter domination is restored and at least the acceleration ends. This would make it possible to define the S-matrix in string theory, which cannot be done in an eternally accelerating universe.

The divergence of perturbations can be related to previous considerations of ghost and other instabilities in Gauss-Bonnet cosmologies[151, 152]. Though as well known, when expanded about a Minkowski spacetime, ghost modes do not appear in Gauss-Bonnet gravity, it has been pointed out that this does not necessarily hold in the FRW background since such is characterized by non-constant curvature. Ghost modes appear when the sign in front of the kinetic term in an equation of motion for a variable to be quantized turns negative<sup>3</sup>. A related instability is due to an effectively imaginary propagation speed in the equation of motion. When such is present in an action for a canonical field, also the solutions of its classical evolution equation could be expected to exhibit divergent behaviour. This far such actions have been derived for scalar modes evolving under Gauss-Bonnet gravity only in a vacuum (i.e. when  $\rho_m = 0$ ). However, in dark energy cosmologies featuring the Gauss-Bonnet term one should take into account both matter and the corrections to Einstein gravity. Application of the vacuum stability (quantum or classical) constraints to these models is a priori inconsistent (except in the case that one considers only an asymptotic situation where the matter density  $\rho_m$  has diluted away). Here we can provide evidence that the same conditions determine the stability of the field in the presence as well as in the absence of other fields. It can be shown that the canonical action for the potential  $\Phi$  in vacuum features an effective propagation speed [155]

$$s_{SC} = \frac{-x^2[4 + \mu(5\mu - 8)] + \mu^2[6(1 + \epsilon)(\mu - 1) + y]}{(\mu - 1)[3\mu^2 - 4(\mu - 1)x^2]}, \quad (6.8)$$

from which we can see that the linearized matter perturbations diverge exactly at the points where this propagation speed changes its sign. This establishes that the classical stability can be, even in presence of dust-like matter (which is the relevant case for

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<sup>2</sup>A case in which this would have happened before the present time would clearly be ruled out. However, in our example model parameter combinations that would lead to this divergence of perturbations before the present day are not cosmologically interesting since they are outside the contours of Fig. 6.1.

<sup>3</sup>Among the effects that have been claimed to possibly occur at macroscopic level are particle overproduction and breaking of causality. Both of these could be observable on cosmological scales, for instance via modification of the structure formation, by detecting gravitational signals from the future or by measuring breaking of Lorentz invariance. However, it is not clear how to assess particle production unless formally quantizing the fields with creation-annihilation operators. On the other hand, causality breaking might occur for superluminal propagation ( $s_{SC} > 1$ , which is different from the instability  $s_{SC} < 0$ ), but the theoretical status of formally superluminal propagation in curved spacetime is not perhaps yet well established[153]. Occasionally it is claimed that such ghost conditions are not relevant if we regard the classical action (2.16) as just an effective field theory and do not insist that the fields involved should allow a consistent quantization[154].

dark energy cosmologies, therefore we leave the generalization to  $w_m \neq 0$  to other studies) deduced from the vacuum action for the canonical gauge-invariant potential  $\Phi$  (or equivalently, from that of  $\Psi$ , for that matter).

Let us also mention that the occurrence of  $w_{eff} < -1$  does not have such a direct relation with these conditions. Therefore it is possible to have phantom universe and avoid a divergence of perturbations in these models, though probably arranging  $w_{eff} < -1$  at some stage would be more difficult if  $s_{SC} > 0$  is required throughout the cosmic evolution. In models where a physical fluid crosses the phantom divide, perturbations tend to diverge, because of  $1/(1+w)$ -terms; but the instability discussed here is different.

## 6.2 Isocurvature quintessence: ISW-LSS correlation

The ISW is due to the net gravitational redshifts of the CMB photons as they travel through time-varying potentials. Because the gravitational potentials are sourced by the local inhomogeneities, the ISW effect is correlated with the evolving large scale structure. The correlation is positive when the structure is decreasing, since a smoothing gravitational well will leave a hotter spot in the CMB.

The cross correlation as a function of the angle  $\theta = |\hat{n}_1 - \hat{n}_2|$  between the directions where CMB and LSS are observed in the sky can be defined as[156]

$$C^X(\theta) = \langle \Delta_{ISW}(\hat{n}_1) \delta_{LSS}(\hat{n}_2) \rangle = \sum_{\ell=2}^{\infty} \frac{2\ell+1}{4\pi} C_{\ell}^X P_{\ell}(\cos \theta). \quad (6.9)$$

In the sum we have Legendre-expanded  $C^X$ , and removed the monopole and dipole contributions which cannot be measured in practice. It follows that the angular cross correlation power spectrum can be written as[157]

$$C_{\ell}^X = 4\pi \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}^2 \tilde{I}_{\ell}^{ISW}(k) \tilde{I}_{\ell}^{LSS}(k), \quad (6.10)$$

where the functions  $I_{\ell}$  are defined as

$$\tilde{I}_{\ell}^{ISW} = \int e^{-\tau(z)} \left( \frac{d\tilde{\Phi}_k}{dz} + \frac{d\tilde{\Psi}_k}{dz} \right) j_{\ell}[kr(z)] dz, \quad (6.11)$$

$$\tilde{I}_{\ell}^{LSS} = b \int \phi(z) \tilde{\delta}_m^k j_{\ell}[kr(z)] dz. \quad (6.12)$$

We denote by tilde perturbation variables which are in units of primordial  $\mathcal{R}$ . A linear bias  $b$  is assumed. The window functions  $\phi_i(z)$  we use are of the form[157]

$$\phi_i(z) = \frac{3}{2\Gamma(\frac{7}{3})z_i^{-3.5}} z^2 e^{-(z/z_i)^{1.5}}. \quad (6.13)$$

This window function probes redshifts of about  $z_i$ , and so in practice  $z_i$  should be smaller than one. Here we will use rather high redshift  $z_i = 0.9$ , since the effects we are looking for turn out occur at large scales (and smaller  $k$  are probed by larger  $z$ ).

As concrete example we will study ISW-LSS correlation a quintessence model. Many mechanisms for suppression of low multipoles of the CMB spectrum have been invented, due to the ostensible lack of power at large scales observed by the WMAP experiment. Though the statistical significance of this observation is questionable, some models it has motivated are interesting also in other respects. One such model is quintessence with anticorrelated isocurvature initial conditions. In the following we will check whether such initial conditions, originally set up in view of the low multipoles of the CMB spectrum, would leave an observable signature in the  $C_\ell^X$  spectrum.

The potential we adopt for the quintessence field is

$$V(\phi) = \frac{1}{2}m_\phi^2\phi^2. \quad (6.14)$$

This potential permits a solution where the EoS of the quintessence field is nearly a constant  $w_Q = -1$  until recently. Then possible initial isocurvature perturbations will not decay but survives at large scales. It has been found that such isocurvature perturbations, if anticorrelated with the comoving curvature perturbation, could lead to suppression of the low multipoles in the CMB spectrum[158]. The suppression occurs because the enhanced late ISW effect cancels adiabatic SW contribution at large scales[159].

We will demonstrate this with a model that has  $h = 0.72$ ,  $\Omega_b = 0.046$ ,  $\Omega_m = 0.27$  and  $\Omega_Q = 0.73$ . It turns out that these cosmological parameters are consistent with the mass scale  $m_Q = 10^{-42}$  GeV for quintessence, when the initial value for the field (at rest) is  $Q_{init} = 3.36M_p$ . We will use scale invariant spectrum for the initial perturbations, with amplitude  $A_s = 0.86$ , and optical depth  $\tau = 0.166$ . We characterize the isocurvature fluctuation by the parameter  $r_Q$ ,

$$r_Q \equiv \frac{\delta Q_{init}}{M_p \mathcal{R}}, \quad (6.15)$$

where  $\mathcal{R}$  denotes primordial value of the comoving curvature perturbation, and  $\delta Q_{init}$  also is gauge-invariant since the field is at rest.

However, ISW-LSS correlation does not seem to have potential to distinguish isocurvature perturbations in this quintessence model. We plot the functions  $I_2(k)$  in Fig.(6.3). There we see that the additional ISW effect in the anticorrelated isocurvature case is best seen at the scales larger than about 2000 Mpc. Going to smaller scales the difference begins to vanish since the isocurvature mode decays away inside the horizon. However, the LSS integral function Eq.(6.12) is peaked at about an order of magnitude smaller scales. Thus the ISW-LSS cross correlation will be similar regardless of the initial conditions for quintessence. There will be a small difference in  $C_2^X$  between the two cases, but it will be negligible in the resulting  $C^X(\theta)$  (which in practice is better suited for comparing to observations, since errors in  $C_\ell^X$  spectrum become large) calculated from Eq.(6.9). For higher  $\ell$ 's, the effect of the isocurvature initial conditions in resulting  $C_\ell^X$ 's will be of course even smaller, since these correspond to smaller angles and thus smaller scales. The  $z_i$  we used in Fig.(6.3) is about the maximum redshift from where one finds usable tracers of the large scale structure[160]. Therefore, when the window Eq.(6.13) is set to such distances, one probes the largest scales that are possible with the ISW-LSS cross correlation. In Fig.(6.4) show the CMB and total matter power spectrum (including CDM and baryons) in this model. The reason that ISW-LSS correlation is insensitive to the isocurvature mode could

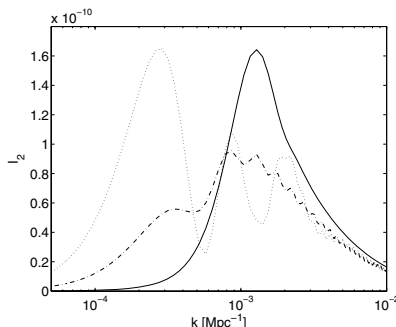


Figure 6.3: Functions  $I_2(k)$ . Dash-dotted line is  $I_2^{ISW}(k)$  when  $r_Q = 0$ , and dotted line the same with  $r_Q = -15$ . The solid line is  $10^{-4}I_2^{LSS}$  (rescaled to fit in the same figure). The extra ISW contribution in the isocurvature case appears at scales much larger than could be probed by the large scale structure.

be anticipated from the matter power spectra in Fig.(6.4), from which we see that the isocurvature initial conditions make difference only at scales comparable to or outside the horizon. These scales cannot be probed by LSS surveys, although they, as seen in Fig.(6.4), contribute significantly to the low multipoles of the CMB spectrum.

To end this subsection, we will briefly mention some other ways than the isocurvature mechanism to reduce the CMB quadrupole and how these could be detected in the ISW-LSS correlation. A simple way to decrease the quadrupole is to introduce a cut-off in the primordial spectrum of perturbations. Since this cut-off would correspond to scales of order 2000 Mpc[159], it does not either leave an imprint in the cross-correlation. One can also get a somewhat lower CMB quadrupole by varying the sound speed of dark energy[159]. That will affect the perturbation evolution also on scales accessible with the ISW-LSS correlation, since when  $c_s^2 \neq 1$  (then the field cannot be a canonical scalar), Jeans scale for dark energy can be lower and dark energy can cluster at smaller scales. Therefore varying the sound speed has an impact on the cross correlation [157], as seen in Fig.(6.5). In the previous section we noted that in models featuring dark energy anisotropic stress it is also possible to reduce the relative power from  $C_2$  in the CMB anisotropies. However, the effect to the correlation spectrum from the anisotropic stress is degenerate with the sound speed when all the three parameters ( $w, c_s^2, c_{vis}^2$ ) are constant. The more general case might be interesting to study in the future.

### 6.3 Gravitational dark energy: Matter power spectrum

In Section (2.3) we highlighted some appealing properties of the first order formalism for modified gravity. In Fig. (3.1) we showed explicitly that within this framework, one may construct a gravitational alternative to dark energy which seems to be allowed by observations (though tight constraints may be imposed on some parameterizations). However, this applies only for the idealization of a perfectly smooth universe. Considering cosmological structures one finds that observations restrict the allowed models in their present

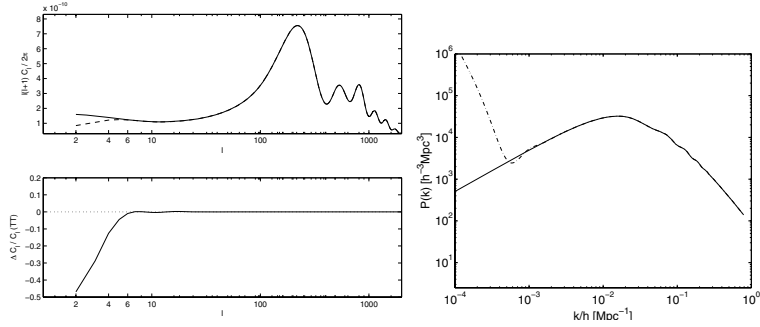


Figure 6.4: Right panel: The CMB anisotropies with  $r_Q = 0$  and with  $r_Q = -15$ . The impact of quintessence isocurvature initial conditions is seen only on the small multipoles. Left panel: Matter power spectra with  $r_Q = 0$  (solid) and with  $r_Q = -15$  (dash-dotted). Isocurvature initial conditions affect only superhorizon scales.

form to a rather tiny modifications of general relativity with a cosmological constant. Here we will briefly outline how one arrives at these results, and then consider possibilities of reincarnating these models in some slightly different forms.

The cosmological perturbation equations of extended gravity in the Palatini formulation [4] feature additional terms due to the curvature corrections. As an example, for an action (2.27) with  $k = 1$  (in the following formulae,  $k$  will denote the wavenumber) the energy constraint Eq.(4.23) is generalized to

$$\begin{aligned}
 \left(2H + \frac{\dot{F}}{F}\right) a\kappa + (6K - 2k^2) \varphi + \frac{1}{2F} \left(-\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F} + 3H\dot{F}\right) \alpha \\
 = \frac{1}{F} \left\{ -a^2\delta\rho - \omega\dot{\phi}\delta\dot{\phi} - \frac{1}{2} \left[ \omega'\dot{\phi}^2 + a^2(2V - f)_{,\phi} \right] \delta\phi \right. \\
 \left. + \left[ 3(H^2 + K^2) - \frac{3\dot{F}^2}{4F^2} - \frac{a^2}{2}R + k^2 \right] \delta F + \left( \frac{3}{2} \frac{\dot{F}}{F} + 3H \right) \delta\dot{F} \right\}.
 \end{aligned} \tag{6.16}$$

We have  $\delta f = F\delta R + f_{,\phi}\delta\phi$ , and similarly  $\delta F = F_{,R}\delta R + F_{,\phi}\delta\phi$ . However, these functions can also be expressed in a form proportional to matter perturbations using the structural equation Eq.(2.35). Consider the example  $f = R - \alpha_0/(3R)$  from Table 2.1. There we have

$$\delta f = \left(1 + \frac{\alpha_0}{3R^2(T)}\right) \left(-\frac{1}{2} - \frac{T}{2\sqrt{T^2 + 4\alpha_0}}\right) (-\delta\rho + 3\delta p), \tag{6.17}$$

$$\delta F = \frac{\alpha_0}{3R^3(T)} \left(1 + \frac{1}{\sqrt{T^2 + 4\alpha_0}}\right) (-\delta\rho + 3\delta p). \tag{6.18}$$

It is straightforward to obtain the derivatives  $\delta\dot{F}$  and  $\delta\ddot{F}$  from Eq.(6.18). In this manner  $\delta R$  can be related to  $\delta T$ , and one can consider the right-hand sides of the perturbed field



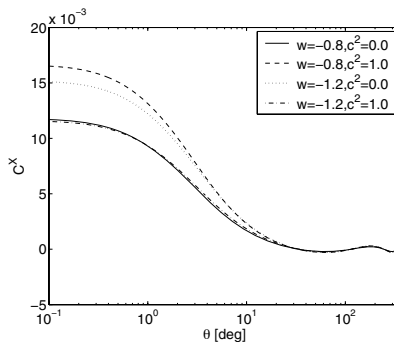


Figure 6.5: Cross correlation, assuming different (constant) dark energy parameters. Variations of the EoS and well as the sound speed affect the cross correlation, though one could not determine both of these parameters from the correlation alone.

equations as matter sources in the modified gravity. However, one possibly appealing gauge condition is to set  $\delta F = 0$ . This eliminates the complicated effective matter source terms in the right hand side, while leaving some  $f$ -dependent terms to modify the evolution of the metric perturbations. In the  $f = f(R)$  case this is the gauge where  $\delta T = 0$ . Since in a dust filled universe this gauge coincides to the uniform density gauge ( $\delta T = -\delta\rho = 0$ ), one can there with minimal effort derive a relatively simple closed form differential equation for the evolution of the velocity perturbation. Switching then to the comoving gauge where the velocity perturbation in turn vanishes while the density fluctuation is nonzero, one finds that comoving gauge matter perturbation is governed by

$$\ddot{\delta}_v = \frac{1}{3FH^2(2FH + \dot{F})} \left\{ -3H \left[ 2FH(FH^2 + \ddot{F}) - 2\dot{F}^2H + \dot{F}F(-2\dot{H} + H^2) \right] \dot{\delta}_v + \left[ 6F^2H^2(\ddot{H} + 2\dot{H}H) + 6\dot{F}^2H(H^2 - \dot{H}) - \dot{F}F(6\dot{H}^2 - 3\ddot{H}H + H^2k^2) + 6\ddot{F}FH(H^2 - \dot{H}) \right] \delta_v \right\}. \quad (6.19)$$

The subindex  $_v$  is there as a reminder of the gauge choice ( $v = 0$ ). When  $K \neq 0$ , the form of this equation is the same, but spatial curvature would of course modify the background evolution.

The curvature corrections induce effective pressure fluctuations in matter, leading to the gradient term in Eq.(6.19), which is nonvanishing except at the  $\Lambda$ CDM limit. While the other deviations from the perturbation evolution in the  $\Lambda$ CDM scenario may be small when  $F$  is close to unity, the gradient can still be large at small enough scales. As one could expect from our discussions in the previous section, inhomogeneities are significantly affected by the additional matter couplings, the most sensitive effect being the response of modified gravity to spatial variations in the distribution of matter. This effect can be used to very severely restrict the allowed parameter space in these models. As an example, in Ref.[6] we presented quantitative constraints for the parameterization A in Eq.(3.18) ensuing from the matter power spectrum alone, showing that these are a couple orders

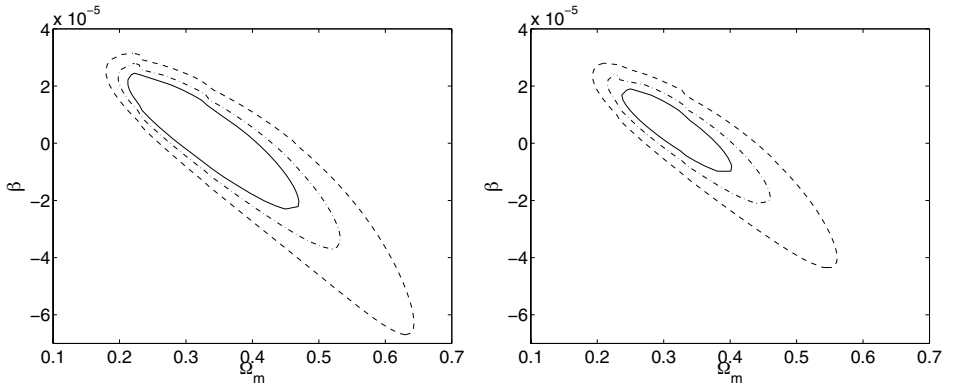


Figure 6.6: The 68, 90 and 99 % confidence contours arising from fitting to the SDSS data. In the left panel the primordial scalar spectral index  $n_S = 1$  and in the right panel  $n_S = 1 \pm 0.2$ . The normalization is arbitrary. The model is constrained very near the  $\Lambda$ CDM one, which corresponds to  $\beta = 0$ .

of magnitude tighter than the background constraints (recall FIG. 3.1). We quote the resulting constraints here in FIG. 6.6.

It would seem difficult to find a way to produce the observed large scale structure in the presence of the modified matter couplings peculiar to these alternative gravity theories. If this could be accomplished by invoking exotic properties to the matter components (isocurvature initial conditions, or may be entropic perturbations to cancel the effects of the couplings), it would not be without adding fine tuning and ad hoc assumptions to these models in analogy with the versions of interacting models discussed in Section 5.1.1.

A better approach might be to find a covariant way to introduce such a scale-dependence on the curvature corrections that their influence would be restricted to the largest cosmological scales. One might wonder if the first order  $f(R)$ -gravities would be better suitable dark energy candidates in the Einstein frame, since as discussed in Section 2 (see also Appendix A) such change of frame is possible to perform and there the physical phenomena will be different (that is, when the change of frame is accompanied with the corresponding identification of physical variables).

Background evolution in such cases has been studied previously[61]. About the perturbations we can say the following. In the Einstein additional matter couplings are manifested in a non-minimal coupling of the scalar field with matter. The somewhat curious scalar field is now devoid of kinetic term: it would not move without the non-minimal coupling. Stayed it still, the model would be equivalent to general relativity (with possibly a  $\Lambda$ -term), as it indeed is unless there is some matter with non-vanishing trace. Thus the deviations from Einstein theory are in this frame due to the effects of the coupling, and when the structure formation is considered, similar features as in the Jordan frame will be seen in clustering of CDM and baryons, now seen as workings of the scalar field. For completeness, we have derived the evolution equation for matter perturbations in the

Einstein frame,

$$\begin{aligned}\ddot{\delta}_v^E &= - \left[ H + \dot{\phi} C'(\phi) \right] \dot{\delta}_v^E \\ &- \left[ 2\dot{H}H - \ddot{H} + \dot{H}\dot{\phi}C'(\phi) + \ddot{\phi}HC'(\phi) + \dot{\phi}^2HC''(\phi) + \frac{\dot{\phi}C'(\phi)k^2}{3} \right] \frac{\delta_v^E}{H},\end{aligned}\quad (6.20)$$

where  $C'(\phi) = F'(\phi)/F(\phi)$ . The limit  $C' = 0$  is the same as  $F = 1$  limit in Eq.(6.19); as expected, the  $\Lambda$ CDM model is recovered there.

Recently another variant of the Palatini  $f(R)$  gravity has been introduced by Carroll *et al* [161], who consider modified-source gravity, which is similar but not equivalent to the Palatini form of  $f(R)$  gravities. There again appears an auxiliary scalar, which is not a true degree of freedom in the sense that it does not have a kinetic term (at least such that could not be integrated away); this scalar can be eliminated in favour of matter density to yield effectively nonlinear matter sources in the RHS of gravitational field equations. In this framework again a new scale-dependent term, absent in general relativity, enters the matter perturbation evolution equations. Carroll *et al* work in the metric formalism but find a scalar-tensor theory which actually features a nondynamical scalar. This comes about as follows. They transform an  $f(R)$  gravity into the Einstein frame and then erase the kinetic term. From our computations in Appendix A we see that this is equivalent to considering the Palatini version of  $f(R)$  gravity in the Einstein frame. Then they transform back into Jordan frame to find a  $F(\phi)R$  scalar-tensor theory (in their case  $F(\phi) = e^{2\phi}$ ) with the kinetic term  $\omega = -3F'^2/(2F)$ . From the equivalence we showed in Ref.[4] one sees such is the same as a Palatini form of the  $F(\phi)$  scalar-tensor theory devoid of kinetic term. Therefore the appearance of the gradient in the perturbation equations is not a surprise.

These examples in mind, one might wonder whether a non-minimal coupling to matter (a feature in the Einstein frame) could be canceled by a non-minimal curvature coupling (a feature of the Jordan frame) when they would be allowed in the play simultaneously (as is the case in any other frame). Let us therefore consider an action with three free functions of the scalar field, a potential  $V(\phi)$ , a curvature coupling  $F(\phi)$  and a matter coupling  $\gamma(\phi)$ :

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R(g_{\mu\nu}) + \frac{3}{4} \frac{F'(\phi)^2}{F} (\partial\phi)^2 - V(\phi) + \gamma(\phi) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right]. \quad (6.21)$$

Metric variation is to be applied to this action; to consider it in the Palatini formalism, one just erases the kinetic term. An algebraic equation of motion follows for the scalar field,

$$V'(\phi) - 2 \frac{F'(\phi)}{F(\phi)} V(\phi) = - \left[ \gamma(\phi) + \frac{1}{2} \frac{F'(\phi)}{F(\phi)} \right] T_m, \quad (6.22)$$

indicating that there exists a second order equation for the evolution of matter perturbations. Deriving it as in the previous cases, one finds that the condition for the disappearance of the gradient there is equivalent to condition that the RHS of Eq.(6.22) vanishes. When it vanishes, the scalar must be constant, and then the action (6.21) becomes general relativity with a cosmological constant. This shows that even in a modified-source gravity more general than some previous models (including them as specific cases) one cannot get

rid of the scale-dependence in cosmological evolution of matter inhomogeneities, except in the limit where the modification becomes trivial and is effectively just a  $\Lambda$ -term.

Previously we referred to an interesting possibility that the matter Lagrangian might also depend on the independent connection that is introduced in the Palatini formulation. We will now briefly review the implications to the theory and speculate that in such a case the structure formation problem might be alleviated. So consider an action

$$S = \int d^n x \sqrt{-g} \left[ \frac{1}{2} f(R(g_{\mu\nu}, \hat{\Gamma}_{\beta\gamma}^\alpha)) + \mathcal{L}_m(g_{\mu\nu}, \hat{\Gamma}_{\beta\gamma}^\alpha, \dots) \right]. \quad (6.23)$$

Now the so called hypermomentum,

$$\Delta_\lambda^{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta \hat{\Gamma}_{\mu\nu}^\lambda} \quad (6.24)$$

is non-zero. Theories where the hatted connection inhabits the matter sector could be called metric-affine  $f(R)$ -theories to distinguish them from the  $f(R)$ -gravities in the Palatini formulation[55]. When writing the field equations for the former, we have also the curvature,

$$f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) = T_{\mu\nu}, \quad (6.25)$$

since the connection determining it is now different from the conformally metric-compatible one. It must be solved from a rather complicated set of equations one finds by varying the action (6.23) wrt to  $\hat{\Gamma}_{\mu\nu}^\lambda$  and using the definition of the hypermomentum (6.24),

$$\frac{1}{\sqrt{-g}} \left[ -\hat{\nabla}_\lambda \left( \sqrt{-g} f'(\hat{R}) g^{\mu\nu} \right) + \hat{\nabla}_\sigma \left( \sqrt{-g} f'(\hat{R}) g^{\sigma(\mu} \delta_{\lambda}^{\nu)} \right) \right] = \Delta_\lambda^{\mu\nu}. \quad (6.26)$$

The usual notion is that metric-affine gravity reduces to general relativity except at microscopic scales. Any vector or tensor field having an action with explicit dependence on the connection, say the Dirac field, will in principle have a non-vanishing hypermomentum and thus induce non-metricity and torsion leading to deviations from general relativity. However, when considered at macroscopic scales in cosmology, most matter is supposed to be accurately described as a perfect fluid. For example, if matter is composed of particles with spin, the spin being assumed to be randomly oriented, and it cancels out when averaged over some volume. However, while this might hold for known matter and when  $f(R) = R$ , in nonlinear gravity it might be worthwhile reconsider the issue. Even if the expectation value is zero for say spin or polarization, fluctuations about the expectation value might be relevant when considering cosmological perturbations. Especially, when corrections to  $f(R) = R$  become significant, the behaviour of the solutions to the hypermomentum field equations (6.26) could be very different than in the metric-affine version of general relativity. In fact it is unclear whether the averaging of the energy density of a particular medium over some coarse-graining scale to obtain a fluid description is legitimate when the effective energy density is a nonlinear function of the conventional energy-momentum tensor. Already in the Palatini formalism of  $f(R)$  gravity we see an instability of the metric at smallest cosmological scales. It is perhaps plausible that when dark matter is modelled taking into account the unvanishing hypermomentum in the metric-affine framework, yet

new effects, though such in the  $f = R$  case would be restricted to microscopic scales, will appear in cosmology also as nonmetricity comes into play. Even then, however, it is not clear if such effects could somehow tend to relax the system of perturbations near general relativity in such a way that the effect of gradients in equations like (6.19) and (6.20) would be suppressed. It remains to be seen whether one could device such a model naturally. A problem is that to calculate anything one would have to guess some parameterization for the dark matter particle; a second problem is that even then the system of field equations (6.25,6.26) seems to be a bit too complicated to tackle.

# Chapter 7

## Summary

In this thesis, we have discussed various physical models of the dark sector, and their observational consequences. Dark energy has been a major issue in cosmology since its introduction almost a decade ago and many different models have been proposed. Here we classified them into three (occasionally overlapping) classes according to which of the basic assumptions of the standard  $\Lambda$ CDM cosmology each model discarded: 1) the first class consisted of physical fluids which violate the strong energy condition by  $w < -1/3$ , 2) the second class was populated by modifications of gravity and finally, in the third class 3) the backreaction of inhomogeneities were blamed for the apparent acceleration. One can observe that a large percentage of these models (especially in the two first classes) are variations of the scalar field scenario. Though never directly observed in Nature, scalar fields are liked because of their simplicity. They have crucial roles in fundamental theories, and in cosmology much of the machinery was already available from previous studies of the early inflation.

We insisted on the key role of perturbations in falsifying some of the proposals and in general distinguishing between different models. Given some background expansion, one can readily pick up several models from each of the three classes to reproduce it (in the class 3), it is more precise to say that the same observations, of e.g. SNIa luminosity-redshift relation, can be produced by different inhomogeneities). Therefore it seems that using just sets of observations probing solely the background expansion one cannot uncover the physics underlying the acceleration. However, let us note that one might of course make decisions between models with identical expansion history based on the required complexity, amount of free parameters and "naturalness" of their values in each model. Still the effects on perturbations, including the implications to the CMB anisotropies and to the formation of large scale structure, are crucial to uncover when assessing the viability of any serious candidate for dark energy. That makes possible to break degeneracies and to considerably reduce the number of possibilities. The most accurate cosmological data is extracted from the CMB sky, which thus usually provides useful constraints on models of dynamical dark energy. However, the primary CMB radiation comes from earlier universe, where dark energy usually is negligible, and thus the most stringent constraints on dark energy in many cases arise from the large scale structure data, as it probes the present inhomogeneities in matter.

In the introduction to the thesis we reviewed some generalized theories of gravitation, basics of linearized cosmology and different approaches to dark energy. We learned that

no compelling alternative to the cosmological constant  $\Lambda$  exists, but that the possibility of dynamics in dark energy is interesting enough to deserve further investigation.

In the thesis work we have studied dark energies from classes 1) and 2). The so called Cardassian expansion served as an example of unified models, which describe dark matter and dark energy as single fluid as well as a parameterization of modified Friedmann equations. However, a drawback of these models is that they lack a proper derivation from a fundamental theory. Nevertheless, in Refs.[1, 7] we investigated the possible CMB and matter power spectra these models can generate.

In Ref.[2] we scrutinized the perturbation evolution when a coupling between dark matter and dark energy is taken into account. There the action principle was well defined as dark energy was represented by a scalar field with a definite potential and coupling to matter. Then it was possible to exclude some models proposed in the literature.

Imperfect dark energy was the topic of Ref.[5]. There a phenomenological description was used to study the impact of late-time anisotropic perturbations to structure formation. We found that shear stress in dark energy can have non-trivial consequences and in general is not excluded by observations,.

We have also been discussing modified gravity as an alternative to fluid-like dark energy. Specifically, the first order formalism of  $f(R)$  gravities has been shown to easily lead to contradiction with observations of the large scale structure of the universe. We discussed basics of extended gravity theories in Ref.[3], specialized to their cosmological applications, in particular at the linear order, in Ref.[4] and finally in Ref.[6] evaluated numerically the matter power spectrum. That showed that the cosmological background expansion in these models must be practically the same as in  $\Lambda$ CDM. However, like with the Cardassian expansion, these models might be impossible to strictly rule out, as they continuously reduce to the  $\Lambda$ CDM in all their predictions.

The scenario we proposed in Ref.[7] seems to exhibit some desirable features as a hybrid dark energy of classes 1) and 2). There a quintessence-like field is coupled to the Gauss-Bonnet invariant. With this perhaps natural extension of general relativity, the acceleration is onset after a scaling epoch, with all the parameters in the model of the Planck scale. The acceleration is also possibly turned off in the future and might go phantom in between.

## Appendix A

# Conformal transformation

Consider the conformally transformed metric  $\tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}$ , where  $\omega^2(x)$  is the conformal factor. The determinants of the metrics are then related by  $\tilde{g} = \omega^{2n}g$ , where  $n$  is the dimension of the spacetime. The Levi-Civita connection of the transformed metric is then given by

$$\tilde{\Gamma}_{\mu\nu}^{\sigma} = \Gamma_{\mu\nu}^{\sigma} + \frac{1}{\omega} [\delta_{\nu}^{\sigma} \nabla_{\mu} \omega + \delta_{\mu}^{\sigma} \nabla_{\nu} \omega + g_{\mu\nu} g^{\sigma\rho} \nabla_{\rho} \omega]. \quad (\text{A.1})$$

This can be used to calculate the Riemann tensor of the conformally transformed metric,

$$\begin{aligned} \tilde{R}_{\sigma\mu\nu}^{\rho} &= R_{\sigma\mu\nu}^{\rho} - 2 \left( \delta_{[\mu}^{\rho} \delta_{\nu]}^{\alpha} \delta_{\sigma}^{\beta} + g_{\sigma[\mu} \delta_{\nu]}^{\alpha} g^{\rho\beta} \right) \frac{1}{\omega} (\nabla_{\alpha} \nabla_{\beta} \omega) \\ &+ 2 \left( 2\delta_{[\mu}^{\rho} \delta_{\nu]}^{\alpha} \delta_{\sigma}^{\beta} + 2g_{\sigma[\mu} \delta_{\nu]}^{\alpha} g^{\rho\beta} + g_{\sigma[\mu} \delta_{\nu]}^{\rho} g^{\alpha\beta} \right) \frac{1}{\omega^2} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega). \end{aligned} \quad (\text{A.2})$$

The corresponding Ricci tensor and Ricci scalar follow by successive contractions,

$$\begin{aligned} \tilde{R}_{\sigma\nu} &= R_{\sigma\nu} - 2 \left[ (n-2) \delta_{\nu}^{\alpha} \delta_{\sigma}^{\beta} + g_{\sigma\nu} g^{\alpha\beta} \right] \frac{1}{\omega} (\nabla_{\alpha} \nabla_{\beta} \omega) \\ &+ 2 \left[ 2(n-2) \delta_{\nu}^{\alpha} \delta_{\sigma}^{\beta} - (n-3) g_{\sigma\nu} g^{\alpha\beta} \right] \frac{1}{\omega^2} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega), \end{aligned} \quad (\text{A.3})$$

$$\tilde{R} = \frac{1}{\omega^2} R - 2(n-1) g^{\alpha\beta} \frac{1}{\omega^3} (\nabla_{\alpha} \nabla_{\beta} \omega) - (n-1)(n-4) g^{\alpha\beta} \frac{1}{\omega^4} (\nabla_{\alpha} \omega) (\nabla_{\beta} \omega). \quad (\text{A.4})$$

What is often needed in practice are the quantities associated with the original metric expressed in terms of the conformally transformed metric. These inverse transformations can be obtained from the previous formulas, and they are

$$R_{\sigma\mu\nu}^{\rho} = \tilde{R}_{\sigma\mu\nu}^{\rho} + 2 \left( \delta_{[\mu}^{\rho} \delta_{\nu]}^{\alpha} \delta_{\sigma}^{\beta} - \tilde{g}_{\sigma[\mu} \delta_{\nu]}^{\alpha} \tilde{g}^{\rho\beta} \right) \frac{1}{\omega} (\tilde{\nabla}_{\alpha} \tilde{\nabla}_{\beta} \omega) + 2 \tilde{g}_{\sigma[\mu} \delta_{\nu]}^{\rho} \tilde{g}^{\alpha\beta} \frac{1}{\omega^2} (\tilde{\nabla}_{\alpha} \omega) (\tilde{\nabla}_{\beta} \omega),$$

$$R_{\sigma\nu} = \tilde{R}_{\sigma\nu} + \left[ (n-2) \delta_{\sigma}^{\alpha} \delta_{\nu}^{\beta} + \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta} \right] \frac{1}{\omega} (\tilde{\nabla}_{\alpha} \tilde{\nabla}_{\beta} \omega) - (n-1) \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta} \frac{1}{\omega^2} (\tilde{\nabla}_{\alpha} \omega) (\tilde{\nabla}_{\beta} \omega),$$

$$R = \omega^2 \tilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \omega (\tilde{\nabla}_{\alpha} \tilde{\nabla}_{\beta} \omega) - n(n-1) \tilde{g}^{\alpha\beta} (\tilde{\nabla}_{\alpha} \omega) (\tilde{\nabla}_{\beta} \omega). \quad (\text{A.5})$$



The Einstein tensor of  $g_{\mu\nu}$  is then

$$\begin{aligned} G_{\mu\nu} &= \tilde{G}_{\mu\nu} + \left[ (n-2)\delta_\mu^\alpha \delta_\nu^\beta - (2-n)\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta} \right] \frac{1}{\omega} \left( \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega \right) \\ &- \frac{1}{2}(n-1)(n-2)\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta} \frac{1}{\omega^2} \left( \tilde{\nabla}_\alpha \omega \right) \left( \tilde{\nabla}_\beta \omega \right). \end{aligned} \quad (\text{A.6})$$

The covariant derivatives with respect to the Levi-Civita connections of the two metrics are related by

$$\nabla_\mu \nabla_\nu f = \tilde{\nabla}_\mu \tilde{\nabla}_\nu f + \left( 2\delta_\mu^{(\alpha} \delta_\nu^{\beta)} - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta} \right) \left( \tilde{\nabla}_\alpha \omega \right) \left( \tilde{\nabla}_\beta f \right), \quad (\text{A.7})$$

and thus

$$\square f = \omega^2 \square f - (n-2)\tilde{g}^{\alpha\beta} \omega \left( \tilde{\nabla}_\alpha \omega \right) \left( \tilde{\nabla}_\beta f \right). \quad (\text{A.8})$$

Consider then a scalar-tensor theory specified by the action

$$\int d^4x \sqrt{-g} \left[ \frac{1}{2}F(\phi)R - \frac{1}{2}k(\phi)(\partial\phi)^2 - V(\phi) + \mathcal{L}_m(g_{\mu\nu}) \right] \quad (\text{A.9})$$

(where we call the kinetic function  $k$  instead of  $\omega$  like in Eq.(2.17), since the latter is now employed to denote the conformal factor). This action is written in the Jordan frame, since the scalar field couples minimally to matter. Performing then conformal transformation with  $\omega^2(x) = M^{-2}F(\phi(x))$  one finds that the Einstein frame action is

$$\int d^4x \sqrt{-\tilde{g}} \left[ \frac{M^2}{2}\tilde{R} - \frac{1}{2}k_E(\phi)(\partial\phi)^2 - V_E(\phi) + \mathcal{L}_E(\tilde{g}_{\mu\nu}, \phi) \right], \quad (\text{A.10})$$

where

$$k_E(\phi) = M^2 \frac{3F'(\phi)^2 + F(\phi)k(\phi)}{2F(\phi)^2}, \quad V_E(\phi) = \frac{M^4 V(\phi)}{F^2(\phi)}, \quad \mathcal{L}_E(\tilde{g}_{\mu\nu}, \phi) = \frac{M^4 \mathcal{L}(\frac{M^2 g_{\mu\nu}}{F(\phi)})}{F^2(\phi)}.$$

If one would like to have a canonic scalar field, then a redefinition is in order:  $\varphi \equiv \int^\phi k_E^{1/2}(\phi') d\phi'$ .

Now it is also easy to see how the celebrated equivalence of nonlinear gravity and a non-minimally coupled self-interacting scalar field theory arises. Introducing an auxiliary field  $\chi$ , we write an action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{2} (f(\chi) + f'(\chi)(R - \chi)) + \mathcal{L}_m \right]. \quad (\text{A.11})$$

Variation wrt  $\chi$  leads to  $\chi = R$ , so this action corresponds to a nonlinear  $f(R)$  gravity. On the other hand, by redefining  $\phi = f'(\chi)$  we get the scalar-tensor action (A.9) with  $F(\phi) = \phi$ ,  $k = 0$  and  $V(\phi) = \chi(\phi)\phi - f(\chi(\phi))$ . In the context of Palatini variation, the  $R$  in the action is given by Eq.(2.32). Therefore in that case the kinetic term of the equivalent scalar-tensor theory does not vanish like in the metric case, but instead  $k(\phi) = -3/\phi$ . The

relation of scalar-tensor theories in the two variational formulations was derived in Ref.[4]. In  $n$  dimensions, one can rescale the kinetic term as

$$k_{metric}(\phi) = k(\phi) - \frac{(n-1)}{(n-2)} \frac{F'^2(\phi)}{F(\phi)}, \quad (\text{A.12})$$

to arrive at exactly the same field equations as in Palatini version of scalar-tensor theory characterized by the kinetic term  $k$ . One sees that a nonlinear gravity in the Palatini formulation cannot be equivalent with another one in the metric formulation (Refs.[162, 163] suggest there exists a "correspondence", but its precise nature is not clear to us).

## Appendix B

### Perturbed FRW metric

In this Appendix we present some formulae which have been used in the derivation of the field equations. Here all covariant derivatives and curvature variables correspond to the metric  $g$ , Eq.(4.2). Although in the bulk of the text we have explicitly denoted such variables as for example  $R(g)$ , we suppress that notation here. These results do not depend on the variational principle which one uses.

The Levi-Civita connection in the FRW spacetime Eq.(4.2) is

$$\begin{aligned}\Gamma_{00}^0 &= H + \dot{\alpha}, & \Gamma_{0i}^0 &= (\alpha - H\beta)_{,i}, & \Gamma_{ij}^0 &= g_{ij}^{(3)} [H(1 - 2\alpha + 2\varphi) + \dot{\varphi}] + (2H\gamma + \dot{\gamma} + \beta)_{|ij}, \\ \Gamma_{00}^i &= (\alpha - H\beta - \dot{\beta})^{,i}, & \Gamma_{0j}^i &= (H + \dot{\varphi})\delta_j^i + \dot{\gamma}^{i|j}, \\ \Gamma_{jk}^i &= \Gamma_{jk}^{i(3)} + g_{jk}^{(3)}(H\beta - \varphi)^{,i} + \delta_j^i\varphi_{,k} + \delta_k^i\varphi_{,j} + \gamma_{|j}^i{}_{k} + \gamma_{|k}^i{}_{j} - \gamma_{|jk}^i.\end{aligned}\tag{B.1}$$

Useful contractions of these are

$$\Gamma_{\mu 0}^\mu = 4H + \dot{\alpha} + 3\dot{\varphi} + \dot{\gamma}^{i|k}{}_{k}, \quad \Gamma_{\mu i}^\mu = \Gamma_{ki}^{k(3)} + \left[ \alpha + 3\varphi + \gamma^{i|k}{}_{k} \right]_{|i}.\tag{B.2}$$

Covariant derivatives of a field  $f(\bar{x}, t) = \bar{f}(t) + \delta f(\bar{x}, t)$  are then

$$\begin{aligned}a^2\nabla^0\nabla_0 f &= -\ddot{\bar{f}} + H\dot{\bar{f}} - \delta\dot{f} + H\delta\dot{f} + \dot{\bar{f}}\dot{\alpha} + 2\alpha(\ddot{\bar{f}} - H\dot{\bar{f}}), \\ a^2\nabla^0\nabla_i f &= \left[ -\delta\dot{f} + H\delta f + \dot{\bar{f}}\alpha \right]_{,i},\end{aligned}\tag{B.3}$$

$$\begin{aligned}a^2\nabla^i\nabla_0 f &= \left[ \delta\dot{f} - H\delta f - \dot{\bar{f}}\beta + \dot{\bar{f}}(\alpha + 2H\beta) \right]^{,i}, \\ a^2\nabla^i\nabla_j f &= -\delta_j^i H\dot{\bar{f}} + \left[ \delta_j^i(2H\alpha - \dot{\varphi}) - \frac{1}{a}\chi^{i|j}{}_{j} \right] \dot{\bar{f}} + \delta f^{i|j}{}_{j} - \delta_j^i H\delta f, \\ a^2\square f &= -\ddot{\bar{f}} - 2H\dot{\bar{f}} - \delta\ddot{f} - 2H\delta\dot{f} + \delta f^{i|k}{}_{k} + (2\ddot{\bar{f}} + H\dot{\bar{f}})\alpha + \dot{\bar{f}}(\dot{\alpha} + a\kappa).\end{aligned}\tag{B.4}$$

The Ricci tensor can be calculated using Eq.(2.2) with Eqs. (B.1),

$$\begin{aligned}
a^2 R_0^0 &= 3\dot{H} - a\dot{\kappa} - 2Ha\kappa + 3(H^2 - \dot{H})\alpha - \Delta\alpha \\
a^2 R_0^i &= 2 \left[ H\alpha - \dot{\varphi} + (\dot{H} - H^2)\beta \right]^{,i}, \\
a^2 R_i^0 &= 2 \left[ -H\alpha + \dot{\alpha} \right]_{,i} \\
a^2 R_j^i &= \delta_j^i (\dot{H} + 2H^2) + R_j^{i(3)} \\
&\quad + \delta_j^i \left[ \ddot{\varphi} - H(\dot{\alpha} - 5\dot{\varphi}) - 2(\dot{H} + 2H^2)\alpha - 2R_j^{i(3)}\varphi + \Delta(-\varphi + \frac{H}{a}\chi) \right] \\
&\quad + \left[ -\alpha - \varphi + \frac{1}{a}\dot{\chi} + \frac{H}{a}\chi \right]_{,j}^{,i}
\end{aligned} \tag{B.5}$$

The curvature scalar is

$$a^2 R = 6(\dot{H} + H^2) + R^{(3)} + \left[ -a\dot{\kappa} - 4Ha\kappa + 3(H^2 - \dot{H})\alpha - (R^{(3)} + 2\Delta)\varphi - \Delta\alpha \right]. \tag{B.6}$$

## B.1 Velocities and viscosities

In our notation the four-velocity  $u_\mu$  is

$$u_\mu = a \left[ -(1 + \alpha), -v_{,i} \right]. \tag{B.7}$$

It follows that

$$u^\mu = \left[ 1 - \alpha, (-v + \beta)^{,i} \right] / a. \tag{B.8}$$

The divergence of four-velocity is then

$$u^\mu_{;\mu} = \left[ 3H(1 - \alpha) + 3\dot{\varphi} - \Delta(v - \frac{1}{a}\chi) \right] / a. \tag{B.9}$$

The projection tensor  $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$  can be evaluated to linear order as

$$h_{\mu\nu} = a^2 \begin{pmatrix} 0 & (v - \beta)_{,i} \\ (v - \beta)_{,i} & g_{ij}^{(3)}(1 + 2\varphi) + \gamma_{ij} \end{pmatrix}. \tag{B.10}$$

Then one can calculate the Navier-Stokes shear viscosity [164]

$$\pi_{\mu\nu} \equiv \varsigma \left( u_{\mu;\alpha} h_\nu^\alpha + u_{\nu;\alpha} h_\mu^\alpha - \frac{2}{3} u^\alpha_{;\alpha} h_{\mu\nu} \right). \tag{B.11}$$

From this definition follows identically that  $\pi_{\mu\nu}$  is traceless, and that the projection  $\pi_{\mu\nu} u^\nu = 0$ . In the perturbed cosmological metric we have specifically that

$$\pi_{0\alpha} = \pi_{\alpha 0} = 0, \quad \pi_{ij} = 2\varsigma a \left( \delta_i^k \delta_j^l - \frac{1}{3} g^{(3)kl} g_{ij}^{(3)} \right) \left( -v + \frac{1}{a}\chi \right)_{|kl}. \tag{B.12}$$

Then the definition

$$(\rho + p)\sigma \equiv -(\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij})\pi^{ij} \quad (\text{B.13})$$

gives us in Fourier space that

$$\sigma = -\frac{8\zeta}{9} \frac{a(k^2 - 3K)}{\rho + p} (v - \frac{1}{a}\chi). \quad (\text{B.14})$$

Thus  $\sigma$  is given by the gradient of the velocity field and the metric shear. By Eqs.(4.9) and (4.11),  $\sigma$  is gauge-invariant. In the synchronous gauge this is written as in Eq.(5.36) in flat space.

The bulk viscosity  $\pi_{\mu\nu}^B \equiv \zeta u^\alpha{}_{;\alpha} h_{\mu\nu}$ , is straightforwardly evaluated using Eqs.(B.10) and (B.9),

$$\pi_{00}^B = 0, \quad (\text{B.15})$$

$$\pi_{0i}^B = \pi_{i0}^B = 3\zeta aH(v - \beta)_{,i}, \quad (\text{B.16})$$

$$\pi_{ij}^B = \zeta g_{ij}^{(3)} a \left[ 3H(1 - \alpha - 2\varphi) + 3\dot{\varphi} - \triangle(v - \frac{1}{a}\chi) \right] + 6aH\gamma_{|ij}. \quad (\text{B.17})$$

Dark energy cosmologies with bulk viscosity have been studied recently, though only viscous modifications to the Friedmann equation. At that level effects of viscous dark energy with some EoS  $w_1(a)$  is equivalent to some other non-viscous dark energy with a different EoS  $w_2(a)$ .

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